ISSN 0019-5588

Indian Journal of Pure & Applied Mathematics

DEVOTED PRIMARILY TO ORIGINAL RESEARCH IN PURE AND APPLIED MATHEMATICS

VOLUME 20/3
MARCH 1989

INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Published monthly by the
INDIAN NATIONAL SCIENCE ACADEMY

Subscriptions :

For India, Pakistan, Sri Lanka, Nepal, Bangladesh and Burma, Contact :

Associate Editor, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002, Telephone : 3311865, Telex : 31-61835 INSA IN.

For other countries, Contact :

M/s J. C. Baltzer AG, Scientific Publishing Company, Wettsteinplatz 10, CH-4058 Basel, Switzerland, Telephone : 61-268925, Telex : 63475.

The Journal is indexed in the Science Citation Index; Current Contents (Physical, Chemical & Earth Sciences); Mathematical Reviews; INSPEC Science Abstracts (Part A); as well as all the major abstracting services of the World.

Indian J. pure appl. Math., 20 (3) : 205-212, March 1989

MATHEMATICAL MODEL OF POPULATION INTERACTIONS WITH DISPERSAL: STABILITY OF TWO HABITATS WITH A PREDATOR

H. El-Owaidy and A. A. Ammar

Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City Cairo, Egypt

(*Received* 21 *April* 1987; *after revision* 26 *July* 1988)

A system of differential equations of dispersion between two populations in habitats separated by a barrier with a predator feeding indiscriminately on these populations is considered. A region in the $\epsilon_1 - \epsilon_2$ plane where equilibrium points exist, is studied. The stability properties of these equilibrium points are investigated.

1. Indroduction

The subject of the effect of dispersal of populations is a topic of considerable ecological interest[1-4]. Holt[4] has considered a two patch model and a migrating predator from an optimal habitat selection point of view. Hasting[3] focused on spatial diffusion but also had a two patch model and showed the stabilizing effect of high dispersal rates.

The view of the problem taken in this paper, as in Freedman and Waltman[1] and Freedman *et al.*[2], is that "pressure" to disperse is given as a (monotone increasing) function of population size but that dispersal is inhibited by the difficulty of leaving the habitat which we, in turn, think of as surmounting a "barrier". It turns out that the more reasonable parameter is inverse barrier strength. This view appeared in Freedman and Waltman[1]. When this (vector) parameter is zero, dispersal is impossible (the barrier is infinite) and each population grows to its carrying capacity.

By the implicit function theorem, Freedman and Waltman[1] showed that, for small values of this parameter, the equilibrium was continuous and they approximated this equilibrium as an expansion in the parameter. This was done in Freedman and Waltman[1] for two habitats and a common barrier strength.

Freedman *et al.*[2] discussed both the two habitats and n-habitats cases each permitted to have a different level of difficulty in its "escape" barrier. In addition, once the population has left its present habitat it may not successfully reach a new one (predation harvesting, or for other reasons). In the analysis, by Freedman *et al.*[2] they regarded the probability of a successful transition between habitats as given and analyzed the question of the existence of the equilibrium as a function of the inverse barrier strength. Under reasonable biological hypotheses and one technical hypothesis, they determined the region exactly for two habitats and, in general case of n habitats.

2. The Model and Equilibria

In this section we shall consider the case where a population is able to disperse among 2-different habitats at some cost to the population in the sense that the probability of survival during a change of habitat may be less than one. This situation is described by a system of two prey and one predator of the form

$$\left.\begin{aligned} x_1' &= \alpha_1 x_1 \left(1 - \frac{x_1}{k_1}\right) - \beta_1 x_1 y - \epsilon_1 x_1 + \epsilon_2 p_{21} x_2 \\ x_2' &= \alpha_2 x_2 \left(1 - \frac{x_2}{k_1}\right) - \beta_2 x_2 y - \epsilon_2 x_2 + \epsilon_1 p_{12} x_1 \\ y' &= y(-\gamma + \delta_1 x_1 + \delta_2 x_2) \end{aligned}\right\} \qquad \ldots(2.1)$$

with $p_{12} + p_{21} \leqslant 1$ and $x_i(0) > 0$, $i = 1, 2$ and $y(0) > 0$.

x_i represents the same population (prey) in the rwo different habitats; y is a predator feeding indiscriminately on the two prey x_1 and x_2; β_1 and β_2 measure the feeding rates of the predator on the two prey x_1 and x_2 respectively; γ is the death rate of the predator; δ_1 and δ_2 the conversion rates of prey to predator; ϵ_1 and ϵ_2 not necessarily small, but positive and represent inverse barrier strength in going out of the first habitat and the second habitat; and p_{ij} is the probability of successful transition from ith habitat to jth habitat (where $i \neq j$).

System (2.1) has been discussed by Freedman and Waltman[1] in the special case $p_{12} = p_{21} = 1$ and $\epsilon_1 = \epsilon_2 = \epsilon$ (the case of a common barrier strength) and for sufficiently small positive ϵ (the case of strong barrier). In our paper ϵ_1 and ϵ_2 are positive but not necessarily small. We seek to find the region in the $\epsilon_1 - \epsilon_2$ plane where an interior equilibrium exists and to determine its stability properties. Also, the system (2.1) is discussed by Freedman *et al.*[2] in general case but without predator.

Let $E^*(\epsilon_1, \epsilon_2) \equiv (x_1^*(\epsilon_1, \epsilon_2), x_2^*(\epsilon_1, \epsilon_2), y^*(\epsilon_1, \epsilon_2))$ denote an equilibrium point in positive octant, if it exists. Then, we have the following theorems:

Theorem 2.1—$E^*(0, 0)$ exists if :

$$\left.\begin{aligned} -\frac{\alpha_2 \gamma}{\beta_2 \delta_2 k_2} &< \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} < \frac{1}{\beta_1 \delta_1 k_1} \\ \text{and} \quad \delta_1 k_1 + \delta_2 k_2 &> \gamma \end{aligned}\right\} \qquad \ldots(2.2)$$

where

$$\left.\begin{aligned} x_1^*(0,0) &= \frac{k_1(\alpha_1 \beta_2 \delta_2 k_2 - \alpha_2 \beta_1 \delta_2 k_2 + \alpha_2 \beta_1 \gamma)}{\alpha_1 \beta_2 \delta_2 k_2 + \alpha_2 \beta_1 \delta_1 k_1}, \\ x_2^*(0,0) &= \frac{k_2(\alpha_2 \beta_1 \delta_1 k_1 - \alpha_1 \beta_2 \delta_1 k_1 + \alpha_1 \beta_2 \gamma)}{\alpha_1 \beta_2 \delta_2 k_2 + \alpha_2 \beta_1 \delta_1 k_1}, \\ y^*(0,0) &= \frac{\alpha_1 \alpha_2 (\delta_1 k_1 + \delta_2 k_2 - \gamma)}{\alpha_1 \beta_2 \delta_2 k_2 + \alpha_2 \beta_1 \delta_1 k_1}. \end{aligned}\right\} \qquad \ldots(2.3)$$

Theorem 2.2—(a) $E^*\ (0, \epsilon_2)$ exists if :

$$\left.\begin{aligned} &\frac{\beta_1\,\delta_1\,k_1}{\beta_2\,\delta_2\,k_2} < \frac{\alpha_1}{\alpha_2}\left[\frac{\delta_1\,k_1}{\gamma} - 1\right] \\ \text{and} \\ &0 < \epsilon_2 < \beta_2\left[\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1} + \frac{\alpha_1\,\gamma}{\beta_1\,\delta_1\,k_1}\right] \end{aligned}\right\} \quad \text{...(2 4)}$$

(b) $E^*\ (\epsilon_1, 0)$ exists if :

$$\left.\begin{aligned} &\frac{\beta_2\,\delta_2\,k_2}{\beta_1\,\delta_1\,k_1} < \frac{\alpha_2}{\alpha_1}\left[\frac{\delta_2\,k_2}{\gamma} - 1\right] \\ \text{and} \\ &0 < \epsilon_1 < \beta_1\left[\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta^2} + \frac{\alpha_2\gamma}{\beta_2\,\delta_2\,k_2}\right] \end{aligned}\right\}. \quad \text{...(2.5)}$$

Theorem 2.3— $E^*\ (\epsilon_1, \epsilon_2)$ exists if either

$$0 < \epsilon_1 < \alpha_1\left(1 - \frac{\gamma}{\delta_1\,k_1}\right) \text{ or } 0 < \epsilon_2 < \alpha_2\left(1 - \frac{\gamma}{\delta_2\,k_2}\right). \quad \text{...(2.6)}$$

Proof of Thorem 2.1—Theorem 2.1 is stated in Freedman and Waltman[1] and its proof is a routine algebraic manipulation for solution of the algebraic system

$$\left.\begin{aligned} &\alpha_1\,x_1\left(1 - \frac{x_1}{k_1}\right) - \beta_1\,x_1\,y = 0 \\ &\alpha_2\,x_2\left(1 - \frac{x_2}{k_2}\right) - \beta_2\,x_2\,y = 0 \\ &\delta_1\,x_1 + \delta_2\,x_2 = \gamma. \end{aligned}\right\} \quad \text{...(2.7)}$$

Proof of Theorem 2.2 : At $\epsilon_1 = 0$, equilibria are solutions of following system of equations :

$$\left.\begin{aligned} &\alpha_1\,x_1\left(1 - \frac{x_1}{k_1}\right) - \beta_1\,x_1\,y + \epsilon_2\,p_{21}\,x_2 = 0 \\ &\alpha_2\left(1 - \frac{x_2}{k_2}\right) - \beta_2\,y - \epsilon_2 = 0 \\ &\delta_1\,x_1 + \delta_2\,x_2 = \gamma. \end{aligned}\right\} \quad \text{...(2.8)}$$

From the second equation, we have

$$\beta_2\,y = \alpha_2 - \epsilon_2 - \frac{\alpha_2\,x_2}{k_2} \quad \text{...(2.9)}$$

and from the third equation, we have

$$x_1 = \frac{\gamma - \delta_2\,x_2}{\delta_1}. \quad \text{...(2.10)}$$

Substituting x_1 and y in the first equation yields :

$$-\alpha_1 \beta_2 k_2 (\gamma - \delta_2 x_2)(\delta_1 k_1 - \gamma + \delta_2 x_2) + \beta_1 \delta_1 k_1 (\gamma - \delta_2 x_2)$$

$$[(\alpha_2 - \epsilon_2) k_2 - \alpha_2 x_2] - \beta_2 k_1 k_2 \delta_1^2 p_{21} \epsilon_2 x_2 = 0,$$

this takes the form

$$ax_2^2 + bx_2 + c = 0$$

where

$$a = [\alpha_1 \beta_2 \delta_2 k_2 + \alpha_2 \beta_1 \delta_1 k_1] \delta_2$$

$$-b = \gamma [\alpha_1 \beta_2 \delta_2 k_2 + \alpha_2 \beta_1 \delta_1 k_1] + \delta_2 k_2 [\alpha_2 \beta_1 \delta_1 k_1 - \alpha_1 \beta_2 \delta_1 k_1 + \alpha_1 \beta_2 \gamma] + \delta_1 k_1 k_2 \epsilon_2 [\beta_2 \delta_1 p_{21} - \beta_1 \delta_2]$$

and

$$c = [\alpha_2 \beta_1 - \alpha_1 \beta_2] \delta_1 k_1 k_2 \gamma + \alpha_1 \beta_2 k_2 \gamma^2 - \beta_1 k_1 k_2 \delta_1 \gamma \epsilon_2.$$

From (2.10) for $x_1 > 0$ must be $x_2 < \gamma/\delta_2$.

Let $f(x_2) \equiv ax_2^2 + bx_2 + c$, then

$$f\left(\frac{\gamma}{\delta_2}\right) = \frac{\gamma^2}{\delta_2^2} a + \frac{\gamma}{\delta_2} b + c.$$

Substituting and simplifying we get

$$f\left(\frac{\gamma}{\delta_2}\right) = -\frac{\beta_2 \delta_1^2 k_1 k_2 p_{21} \epsilon_2}{\delta_2} < 0.$$

And $f(0) = c$. Thus $0 < x_2 < \gamma/\delta_2$. ...(2.11)

If $c > 0$, that is if

$$\epsilon_2 < \beta_2 \left(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1} + \frac{\alpha_1 \gamma}{\beta_1 \delta_1 k_1}\right). \qquad ...(2.12)$$

From (2.9), we have

$$\beta_2 k_2 y = (\alpha_2 - \epsilon_2) k_2 - \alpha_2 x_2$$

$$> (\alpha_2 - \epsilon_2) k_2 - \frac{\alpha_2 \gamma}{\delta_2} \text{ by (2.11)}$$

$$> \alpha_2 k_2 - \frac{\alpha_2 \gamma}{\delta_2} - \beta_2 k_2 \left(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1} + \frac{\alpha_1 \gamma}{\beta_1 \delta_1 k_1}\right)$$

$$= \frac{\alpha_1 \beta_2 k_2}{\beta_1} - \frac{\alpha_1 \beta_2 k_2 \gamma}{\beta_1 \delta_1 k_1} - \frac{\alpha_2 \gamma}{\delta_2}.$$

Thus, for $y > 0$, must be

$$\frac{\alpha_1}{\beta_1} > \gamma \left[\frac{\alpha_1}{\beta_1 \delta_1 k_1} + \frac{\alpha_2}{\beta_2 \delta_2 k_2} \right]$$

or

$$\frac{\beta_1 \delta_1 k_1}{\beta_2 \delta_2 k_2} < \frac{\alpha_1}{\alpha_2} \left[\frac{\delta_1 k_1}{\gamma} - 1 \right].$$

This completes the proof of (a), and the proof of (b) is analogus.

Proof of Theorem 2.3

This is the case where $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Equilibria are solutions of the following system of equations:

$$\left.\begin{aligned} &\alpha_1 x_1 \left(1 - \frac{x_1}{k_1}\right) - \epsilon_1 x_1 + \epsilon_2 p_{21} x_2 = \beta_1 x_1 y \\ &\alpha_2 x_2 \left(1 - \frac{x_2}{k_2}\right) - \epsilon_2 x_2 + \epsilon_1 p_{12} x_1 = \beta_2 x_2 y \\ &\delta_1 x_1 + \delta_2 x_2 = \gamma. \end{aligned}\right\} \quad \ldots(2.13)$$

Substituting the second and the third equations in the first equation of system (2.13) and simplifying we have

$$\beta_2 x_2 \left[\frac{\alpha_1}{\delta_1} (-\gamma \delta_2 x_2) - \frac{\alpha_1}{\delta_1^2 k_1} (\gamma - \delta_2 x_2)^2 - \frac{\epsilon_1}{\delta_1} (\gamma - \delta_2 x_2) \right.$$

$$\left. + \epsilon_2 p_{21} x_2 \right] = \frac{\alpha_2 \beta_1}{\delta_1 k_2} x_2 [k_2 - x_2] [\gamma - \delta_2 x_2] - \frac{\epsilon_2 \beta_1}{\delta_1} x_2$$

$$(\gamma - \delta_2 x_2) + \frac{\epsilon_1 \beta_1 p_{12}}{\delta_1^2} (\gamma - \delta_2 x_2)^2.$$

After simplification, we get the form

$$a x_2^3 + b x_2^2 + c x_2 + D = 0$$

where

$$a = \frac{\alpha_1 \beta_2 \delta_2}{k_1} + \frac{\alpha_2 \beta_1 \delta_1}{k_2}$$

$$b = \alpha_1 \beta_2 \delta_1 + \beta_1 \delta_1 \epsilon_2 + \beta_1 \delta_2 p_{12} \epsilon_1 - \beta_2 \delta_1 \epsilon_1 - \alpha_2 \beta_1 \delta_1$$

$$- \frac{2\alpha_1 \beta_2 \gamma}{k_1} - \frac{\alpha_2 \beta_1 \delta_1 \gamma}{\delta_2 k_2} - \frac{\beta_2 \delta_1^2 p_{21} \epsilon_2}{\delta_2}$$

$$c = -\frac{\alpha_1 \beta_2 \delta_1 \gamma}{\delta_2} + \frac{\alpha_2 \beta_1 \delta_1 \gamma}{\delta_2} + \frac{\beta_2 \delta_1 \gamma \epsilon_1}{\delta_2} + \frac{\alpha_1 \beta_2 \gamma^2}{\delta_2 k_1}$$

$$- \frac{\beta_1 \delta_1 \gamma \epsilon_2}{\delta_2} - 2\beta_1 \gamma p_{12} \epsilon_1$$

and

$$D = \frac{\beta_1 \gamma^2 p_{12} \epsilon_1}{\delta_2}.$$

Let

$$f(x_2) \equiv a x_2^3 + b x_2^2 + c x_2 + D.$$

Then, we have :

$$f(0) = D > 0$$

and

$$f\left(\frac{\gamma}{\delta_2}\right) = \frac{\gamma^3}{\delta_2^3} a + \frac{\gamma^2}{\delta_2^2} b + \frac{\gamma}{\delta_2} c + D.$$

Substituting and simplifying, we have

$$f\left(\frac{\gamma}{\delta_2}\right) = -\frac{\beta_2 \delta_1^2 \gamma^2 p_{21} \epsilon_2}{\delta_2^3} < 0.$$

Then, $0 < x_2 < \gamma/\delta_2$. Similarly, $0 < x_1 < \gamma/\delta_1$.

From the second equation of system (2.13), we have :

$$\beta_2 x_2 y = \left[\alpha_2 - \epsilon_2 - \frac{\alpha_2}{k_2} x_2\right] x_2 + \frac{\gamma p_{12} \epsilon_1}{\delta_1} - \frac{\delta_2 p_{12} \epsilon_1}{\delta_1} x_2$$

$$> \left[\alpha_2 - \epsilon_2 - \frac{\alpha_2 \gamma}{\delta_2 k_2}\right] x_2 + \frac{\gamma p_{12} \epsilon_1}{\delta_1} - \frac{\delta_2 p_{12} \epsilon_1}{\delta_1} \frac{\gamma}{\delta_2}$$

$$= \left[\alpha_2 - \epsilon_2 - \frac{\alpha_2 \gamma}{\delta_2 k_2}\right] x_2.$$

Then, $\beta_2 y > \alpha_2 - \epsilon_2 - \frac{\alpha_2 \gamma}{\delta_2 k_2}$. Thus, for $y > 0$, must be

$$\epsilon_2 < \alpha_2 \left[1 - \frac{\gamma}{\delta_2 k_2}\right].$$

Similarly, $\epsilon_1 > \alpha_1 \left[1 - \frac{\gamma}{\delta_1 k_1}\right]$. This completes the proof of Theorem 2.3.

3. Stability

Having established the existence of an equilibrium $E^*(\epsilon_1, \epsilon_2)$, we proceed to examine its stability properties. In fact we have the following theorems :

Theorem 3.1—$E^*(0,0)$ is asymptotically stable.

Theorem 3.2—(a) $E^*(0, \epsilon_2)$ is asymptotically stable if $\beta_1 \geqslant \beta_2$:

(b) $E^*(\epsilon_1, 0)$ is asymptotically stable lf $\beta_2 > \beta_1$.

Theorem 3.3—$E^*(\epsilon_1, \epsilon_2)$ is asymptotically stable if $\beta_1 = \beta_2$.

Proof : The first step is to compute the variational matrix of $E^*(\epsilon_1, \epsilon_2)$, which takes the form

$$V(\epsilon_1, \epsilon_2) = \begin{bmatrix} \alpha_1\left(1 - \dfrac{x_1}{k_1}\right) - \beta_1 y - \epsilon_1 - \dfrac{\alpha_1 x_1}{k_1} & \epsilon_2 p_{21} & -\beta_1 x_1 \\ \epsilon_1 p_{12} & \alpha_2(1-(x_2/k_2)) - \beta_2 y - \epsilon_2 - \dfrac{\alpha_2 x_2}{k_2} & -\beta_2 x_2 \\ \delta_1 y & \delta_2 y & 0 \end{bmatrix} \quad \ldots(3.1)$$

Making use of the system (2.13), the variational matrix takes the form

$$V(\epsilon_1, \epsilon_2) = \begin{bmatrix} -\epsilon_2 p_{21}\dfrac{x_2}{x_1} - \dfrac{\alpha_1 x_1}{k_1} & \epsilon_2 p_{21} & -\beta_1 x_1 \\ \epsilon_1 p_{12} & -\epsilon_1 p_{12}\dfrac{x_1}{x_2} - \dfrac{\alpha_2 x_2}{k_2} & -\beta_2 x_2 \\ \delta_1 y & \delta_2 y & 0 \end{bmatrix} \quad \ldots(3.2)$$

The corresponding characteristic equation is given by

$$\left[\lambda + \frac{\alpha_1 x_1}{k_1} + \epsilon_2 p_{21}\frac{x_2}{x_1}\right]\left[\lambda\left(\lambda + \frac{\alpha_2 x_2}{k_2} + \epsilon_1 p_{12}\frac{x_1}{x_2}\right) + \beta_2\delta_2 x_2 y\right]$$
$$- \epsilon_1 p_{12}[\epsilon_2 p_{21}\lambda - \beta_1\delta_2 x_1 y] + \delta_1 y[\epsilon_2 p_{21}\beta_2 x_2 + \beta_1 x_1$$
$$\times\left(\lambda + \frac{\alpha_2 x_2}{k_2} + \epsilon_1 p_{12}\frac{x_1}{x_2}\right)\Big] = 0. \quad (3.3)$$

Equation (3.3) can be written in the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$a_1 = \frac{\alpha_1 x_1}{k_1} + \frac{\alpha_2 x_2}{k_2} + \epsilon_1 p_{12}\frac{x_1}{x_2} + \epsilon_2 p_{12}\frac{x_2}{x_1}$$

$$a_2 = \frac{\alpha_1\alpha_2 x_1 x_2}{k_1 k_2} + \beta_1\delta_1 x_1 y + \beta_2\delta_2 x_2 y + \frac{\epsilon_1\alpha_1 p_{12} x_1^2}{k_1 x_2}$$

(*equation continued on p. 212*)

$$+ \frac{\epsilon_2 \, \alpha_2 \, p_{21} \, x_2^2}{k_2 \, x_1}.$$

and

$$a_3 = \beta_1 \, \delta_1 \, x_1 \, y \left[\frac{\alpha_2 \, x_2}{k_2} + \epsilon_1 \, p_{12} \, \frac{x_1}{x} \right] + \beta_2 \, \delta_2 \, x_2 \, y \left[\frac{\alpha_1 \, x_1}{k_1} \right.$$

$$\left. + \epsilon_2 \, p_{12} \, \frac{x_2}{x_1} \right] + \epsilon_1 \, \beta_1 \, \delta_2 \, p_{12} \, x_1 \, y + \epsilon_2 \, \beta_2 \, \delta_1 \, p_{21} \, x_2 \, y.$$

Since both a_1 and a_3 are positive, then utilizing the Rough-Hurwize criteria, we get $E^*(\epsilon_1, \epsilon_2)$ is asymptotically stable if $a_1 \, a_2 - a_3 > 0$. After simplification we have:

$$a_1 \, a_2 - a_3 = a_1 \left[\frac{\alpha_1 \, \alpha_2 \, x_1 \, x_2}{k_1 \, k_2} + \frac{\epsilon_1 \, \alpha_1 \, p_{12} \, x_1^2}{k_1 \, x_2} + \frac{\epsilon_2 \, \alpha_2 \, p_{21} \, x_2^2}{k_2 \, x_1} \right]$$

$$+ \delta_1 \, y \left[\frac{\alpha_1 \, \beta_1 \, x_1^2}{k_2} + (\beta_1 - \beta_2) \, \epsilon_2 \, p_{21} \, x_2 \right]$$

$$+ \delta_2 y \left[\frac{\alpha_2 \beta_2 x_2^2}{k_2} + (\beta_2 - \beta_1) \, \epsilon_1 \, p_{12} \, x_1 \right]. \qquad \ldots(3.4)$$

From (3.4), it is clear that if $\epsilon_1 = \epsilon_2 = 0$, then $a_1 \, a_2 = a_3 > 0$ and hence $E^*(0, 0)$ is asymptotically stable. This proves Theorem 3.1.

Also $a_1 \, a_2 - a_3 > 0$ if $\epsilon_1 = 0$ and $\beta_1 \geqslant \beta_2$ or $\epsilon_2 = 0$ and $\beta_2 \geqslant \beta_1$ This proves Theorem 3.2.

Finally, $a_1 \, a_2 - a_2 > 0$ if $\beta_1 = \beta_2$ for every positive values of ϵ_1 and ϵ_2 and this proves Theorem 3.3.

Corollary —From the proofs of Theorems 2.3 & 3.3 it is easy to deduce the ultimate sizes of the two populations when $E^*(\epsilon_1, \epsilon_2)$ is asymptotically stable; for the prey x_i $(i = 1, 2)$ must be $0 < x_i < \gamma/\delta_i$ where $\delta_1 \, x_1 + \delta_2 \, x_2 = \gamma$ that is x_1, x_2 belong to the open straight line segment AB where A $(\gamma/\delta_{1,0})$ and $B(0, \gamma/\delta_2)$ Finally, for the predator y must be

$$y > \frac{1}{\beta} \max \left(\alpha_1 \, \epsilon_1 - \frac{\alpha_1 \, \gamma}{\gamma_1 k_1} , \; \alpha_2 - \epsilon_2 - \frac{\alpha_2 \, \gamma}{\delta_2 \, k_2} \right)$$

REFERENCES

1. H I. Freedman and P. Waltman, *SIAM J. Appl. Math.* 32 (1977). 631-48.
2. H. I. Freedman, B. Rai and P. Waltman, *Technical Report No* 11, Department of Mathematics and Computer Science, EMORY University, Atlanta, Georgia, U.S.A. 1981.
3. A. Hastins, *J. Math. Biol.* 16 (1982), 44-55.
4. R. D. Holt, "Population Dynamics is a Two Patch Environment:; Some Anomalous Conseqences of Optional Habitat Seletion (prepint).

Indian J. pure appl. Math., 20 (3) : 213-228, March 1989

ON CONTROLLABILITY OF NONLINEAR SYSTEMS WITH DISTRIBUTED DELAYS IN THE CONTROL

JERRY U. ONWUATU

Federal University of Technology, SEMNS/Industrial Mathematics
Owerri, Nigeria

(*Received* 4 *August* 1987; *after revision* 1 *August* 1988)

It is known that most natural applications give rise to mechanisms of indirect actions, where the decisions in the control function u are shifted, twisted or combined before affecting the evolution. An example of this delayed action is the model defined by

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s).$$

We give sufficient conditions for the null controllability of such systems when the controls are limited in the sense that they lie in a closed unit ball with zero in its interior. These extend known results.

1. Introduction

Consider the nonlinear control system with distributed delays given by

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) + f(t, x(t), u(t)) \qquad \text{...(1.1)}$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \text{...(1.2)}$$

where $x \in E^n$, $u \in IB \equiv L_\infty([a, b], E^m)$, the space of functions measurable and essentially bounded on finite intervals. It is assumed that the system (1.1) (= (1.2)) has a unique solution $x = x(t) = x(t, t_0, x_0, u)$, for each admissible control $u \in IB$, and each initial state $x(t_0) = x_0$.

The controllability of system (1.1) has been studied by Klamka[4,5], where he showed that under certain conditions on the matrix functions $A(t, x, u)$, $f(t, x, u)$, $B(t, s)$, system (1.1) is globally relatively controllable if

$$\inf_{z\, u \in C_{nm}[t_0, t_1]} \det W(t_0, t_1; z, v) \geq c.$$

Here

$$W(t_0, t_1; z, v)$$

is the controllability matrix (to be defined later). In this method, fixing the variable arguments of the matrix A and the function f by $(z, v) \in C_{nm}[t_0, t_1]$, he arrived at the linear dynamical system

$$\dot{x}(t) = A(t, z(t), v(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s)$$

$$+ f(t, z(t), v(t)).$$

He then applied Schauder's fixed point theorem to this linear system to obtain conditions for global controllability of (1.1).

In our own case we shall treat the equivalent system of (1.1) given by (1.2), where $g(t, x(t), u(t))$ is nonlinear. We shall employ the Alek seev-type variation of constant formula given by Khanh[2] to obtain an integral equation for system (I.2). We shall then use this integral equation to derive necessary and sufficient conditions for the null controllability with constraints for the linear variational system of (1.2). With this result, we shall then provide sufficient criteria for the null controllability of (1.2) with constraints. Indeed, we shall show that if

(i) $g(t, 0, 0) = 0$;

(ii) the system

$$\dot{x}(t) = A(t, x(t)) + B(t)\, u(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \ldots(1.3)$$

is controllable,

where $A(t) = D_2\, g(t, 0, 0)$; $B(t) = D_3\, g(t, 0, 0)$;

(iii) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad \ldots(1.4)$$

is uniformly asymptotically stable, so that the solution of (1.4) satisfies $| x(t) | \leqq k\, | x_0 |^{-\alpha(t-t_0)}$, $\alpha > 0$, $t > 0$, then (1.2) is null controllable with constraints provided g satisfies all smoothness conditions for the existence and uniqueness of solutions.

We shall further apply our results specifically to the systems

$$\dot{x}(t) = g(x(t), u(t)) + \sum_{i=1}^{p} C_i\, u(t - h_i) \qquad \ldots(1.5)$$

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{p} C_i\, u(t - h_i) \qquad \ldots(1.6)$$

where

$$A = D_1\, g\,(0, 0);\; B = D_2\, g\,(0, 0). \qquad \text{...(1.7)}$$

Our results will show that if

(i) $g\,(0, 0) = 0$

(ii) rank $[B, AB, \ldots\ldots, A^{k-1}\, B, \ldots, \ldots\ldots, A^{k-1}\, C_k] = n$

and

(iii) the system

$$\dot{x}\,(t) = g\,(x\,(t), 0) \qquad \text{...(1.8)}$$

is uniformly asymptotically stable then (1.5) is null controllable with constraints.

Sebakhy and Bayounmi[7] considered the system

$$\dot{x} = A\,(t)\,x\,(t) + B\,(t)\,u\,(t) + C\,(t)\;u\,(t - h).$$

This system is totally linear. Their results show that this system is null controllable if and only if

$$\text{rank}\;\Gamma\,(t_0\; t_1) = n$$

where $\Gamma\,(t_0, t_1)$ is the controllability Grammian. This result cannot be applied to systems of the form

$$\dot{x}\,(t) = g\,(t, x\,(t)) + B\,(t)\,u\,(t) + g\,(t)\,X\,(t)$$

where g and f are nonlinear. Thus, our results complements and extends those of Sebakhy and Bayounmi[7].

Khanh[2] considered the nonlinear system

$$\dot{x}\,(t) = g\,(t, x, u\,(t)) + B\,(t, x\,(t))\,u\,(t) + f\,(t, x\,(t), u\,(t)).$$

nonlinear though, but without control delays. Thus, our results extend those of Khanh[2] to include systems with delayed controls. However, our results are of a special nature in the sense that it considers controllability to the origin of E^n, a results that is basic in the study of optimal control problems. Klamka[3,4] considered the following systems

$$\dot{x} = A\,(t)\,x\,(t) + \int_{-h}^{0} [dH\,(t, s)]\,u\,(t + s)$$

a linear system, and

$$\dot{x}\,(t) = A\,(t, x\,(t))\,x\,(t) + \int_{-h}^{0} (d_s\, H\,(t, s)\,x\,(s))\,u\,(t + s)$$

nonlinear. He gave sufficient conditions for controllability of the above systems when the controls are unlimited. Our result specialises the results of the above to controllability to the origin of E^n with limited controls i.e. with constrained controls. Yamatoto[8] and Dauer[1] gave results for the complete controllability of quasi-linear systems of the form

$$\dot{x} = A(x, u, t) X + B(x, u, t) u + f(x, u, t).$$

when the controls are unlimited. Our result strengthens and extends the above results to include systems with control delays.

In section 2, we give the basic notations and define the system equations. Sections 3 gives the controllability and stability theorems for both the linear variational system and the nonlinear system (1.2).

Chapter 4 deals with the applications of the results to system (1.5).

Throughout the paper, we exploit the properties of the 'Reachable' sets in deriving our results.

2. Notations and Preliminaries

Let E denote the real line and $J = [t_0, t_1]$, an interval in E. For a positive integer n, we denote by E^n the space of real n-tuples with the Euclidean norm denoted by $| . |$. If J is any interval of E, the usual Lebesgue space of functions, measurable and essentially bounded from J to E^n will be denoted by $L_\infty(J, E^n)$. M_{nm} will be used for the collection of all real $n \times m$ matrices with a suitable norm. Let $h > 0$ be given. For functions $u : [t_0 - h, t_0] \to E^n$, $t \in [t_0, t_1]$, we use u_t to denote the function on $[-h, 0]$ defined by $u_t(s) = u(t + s)$, for $s \in [-h, 0]$.
We shall consider the nonlinear system

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) + f(t, x(t), u(t)) \quad \ldots(2.1)$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \quad \ldots(2.2)$$

satisfied almost everywhere on $[t_0, t_1]$, where the integral is in Lebesgue-Steiltjes sense with respect to s, $x(t) \in E^n$, $u \in L_\infty([t_0, t_1], E^m)$, $B(t, s)$ is an $n \times m$ matrix-valued function absolutely continuous in s for each fixed t, and of bounded variation in s on $[-h, 0]$ for each $t \in [t_0, t_1]$. We shall assume that $g(t, x(t), u(t))$ (nonlinear in general) is continuous with respect to its arguments and continuously differentiable with respect to x. We shall also assume toat the functions g, g_x, B, are continuous regarding the arguments.

Throughout the sequel, the controls of interest are $IB = L_\infty([t_0, t_1], E^m)$ $IU \subseteq L_\infty([t_0, t_1], E^m)$, a closed and bounded subset of $1B$ with zero in its interior relative to $1B$.

Definition 2.1— The pair $z_t = \{x(t), u_t\}$ is said to be the complete state of the system (2.1) or (2.2).

Definition 2.2— System (2.1) is relatively controllable (or controllable) on $[t_0, t_1] = J$, if for every complete state z_t and every vector x_0, x_1 in E^n there exists a control $u(t)$ defined on $[t_0, t_1]$, such that the corresponding trajectory of system (2.1) satisfies the condition $x(t_1) = x_1$, with $x(t_0) = x_0$. Following Sebakhy and Bayounmi[7], we give the following definition of null controllability.

Definition 2.3— System (2.1) is said to be null controllable at $t = t_1$, if for any initial state $\{x_0, u_{t_0}\}$ on $[t_J - h, t_0]$, there exists and admissible control $u(t) \in IB$, defined on $[t_0, t_1 - h]$ such that the response of the system is brought so the origin of E^n at $t = t_1$ using the control effort

$$u(t) = \{u(t) \text{ on, } [t_0, t_1 - h]\}$$
$$= 0 \text{ on, } [t_1 - h, t_1].$$

That is if given the initial state $\{x_0, u_{t_0}\}$ on $[t_0 - h, t_0]$ there exists a control $u(t) \in IB$ defined on $[t_0, t_1 - h]$ such that $x(t_1) = 0$.

System (2.1) is null controllable with constraints at $t = t_1$ if for any initial state x_0, u_{t_0} on $[t_0 - h, t_0]$, there exists an admissible control $u(t) \in IU$, defined on $[t_0, t_1 - h]$ such that the response $x(t)$ satisfies $x(t_1) = 0$, using the control

$$u(t) = \begin{Bmatrix} u(t) \in IU \text{ on } [t_0, t_1 - h], \\ 0 \in IU, \text{ on } [t_1 - h, t_1]. \end{Bmatrix}$$

Let the system

$$\dot{x}(t) = g(t, x(t), u(t)) \qquad \ldots(2.3)$$

have a unique solution $x = G = G(t, t_0, u, y)$, for each admissible control $u \in IB$ and each initial state $G(t_0, t_0, y, u) = y \in E^n$. If we now set

$$F(t, t_0, y, u) = G_y(t, t_0, y, u) \qquad \ldots(2.4)$$

and

$$K(t, x, u) = g_y(t, x, u) \qquad \ldots(2.5)$$

then it is not difficult to see that the following relations hold for the $n \times n$ matrix functions

$$F_t(t, s, y, u) = K(t, G(t, s, y, u), u(t))\, F(t, s, y, u);\ F(t_0, t_0, u, y) = I_{nn}$$

$$F(t, t_0, u, y(t_0)) = I_{nn} + \int_{t_0}^{t} K(s, G(s, t_0, u, y(t_0), u(s))$$

$$F(s, t_0, y(t_0), u(s))\, ds. \qquad ...(2.6)$$

Thus, Alekseev-type variation of parameter formula for (2.2) (= (2.1)) is given via the result of Khanh[2] by

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, x(s), u(s))$$

$$\left[\int_{-h}^{0} (d_s B(t, s)\, u(t + s))\, ds\right]. \qquad ...(2.7)$$

When we consider the solution $x(t)$ of (2.2) given in (2.7), we note that the values of the control $u(t)$ for $t \in [t_0 - h, t_0]$ enter into the definition of the complete state $z(t_0)$. Thus, the last term of (2.7) must be transformed to take care of this by interchanging the order of the integration. Now, using the unsymmetric Fubini theorem (Klamka[4]), we have the following :

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} dB_s \left(\int_{t_0+s}^{t_0} F(t, l - s, x(l - s),\right.$$

$$u(l - s))\, B(l - s, s)\, u_{t_0}\, dl) + \int_{-h}^{0} d_{B_s}$$

$$\left(\int_{t_0}^{t+s} F(t, l - s, x(l - s), u(l - s)\, B(l - s, s)\, u(l)\, dl\right. \qquad ...(2.8)$$

where the symbol d_{B_s} denotes that the integration is in the Lebesque-Steiltges sense with respect to the variable s in the function $B(t, s)$. If we denote by

$$B_t(l, s) = \begin{Bmatrix} B(l, s) \text{ for } l \leq t, s \in E \\ 0 \quad\quad \text{for } l > t, s \in E \end{Bmatrix} \qquad ...(2.9)$$

then (2.8) yields

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} d_{B_s} \left(\int_{t_0+s}^{t_0} F(t, l-s, x(l-s), u(l-s)).\right.$$

$$B(l - s, s)\, u_{t_0}\, dl) + \int_{-h}^{0} d_{B_s} \left(\int_{t_0}^{t} F(t, l - s, x, u)\, B_t(l - s, s)\, u(l)\, d(l)\right. \qquad ...(2.10)$$

Using the unsymmetric Fubini Theorem, (2.10) can be written in a more convinient form as follows :

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} d_{B_s} \Big(\int_{t_0+s}^{t_0} F(t, l - s, x, u)$$

$$(B(l - s, s)\, u_{t_0}\, dl) + \int_{t_0}^{t} \int_{-h}^{0} F(t, l - s, x, u)$$

$$d_s\, B_t(l - s, s))\, u(l)\, dl. \qquad \ldots(2.11)$$

We now define the $n \times n$ controllability matrix of (2.2) (= (2.1)) by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Big(\int_{-h}^{0} F(t_1, l - s, x, u)\, d_s\, B_{t_1}(l - s, s)\Big)$$

$$\Big(\int_{-h}^{0} F(t_1, l - s, x, u)\, d_s\, B_{t_1}(l - s, s)\Big)^T dl. \qquad \ldots(2.12)$$

Where the symbol T denotes the matrix transpose. Note that $W(t_0, t_1)$ is symmetric and non-negative-definite.

Definition 2.4— The reachable set of (2.2) at time t_1 using L_∞ control is the subset of E^n given by

$$IP(t_1, t_0) = \Big[\int_{t_0}^{t_1} F(t, s, x(s), u(s)) \int_{-h}^{0} (d_s\, B(t, s)\, u(t + s))\, ds : u \in IB\Big].$$

In a similar manner, we define the constraint reachable set of (2.2) as

$$IR(t_1, t_0) = \Big[\int_{t_0}^{t_1} F(t, s, x(s), u(s)) \Big[\int_{-h}^{0} (d_s\, B(t, s)\, u(t + s)\, ds : u \in IU\Big].$$

3. Controllability Theorems

Consider the system

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s\, B(t, s))\, u(t + s) \qquad \ldots(3.1)$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s\, B(t, s))\, u(t + s). \qquad \ldots(3.2)$$

Lemma 3.1— For each fixed $(t_0, s_0, x_0, u_0) \in E^+ \times E^+ \times E^n \times L_\infty$, let $\bar{D}x(t_0, s_0, x_0, u_0)$ denote the partial derivatives of x with respect to its last two arguments at the point (t_0, s_0, x_0, u_0). Then, for every $(h, u) \in E^n \; x \; L_\infty$, we have

$$\bar{D}(t_0, s_0, x_0, u_0)(h, u) = \lambda(t_0, s_0, x_0, u_0, h, u),$$

where the mapping $t \to \lambda(t, s_0, x_0, u_0, h, u)$ of E into E^n is the unique absolutely continuous solution of the linear differential equation

$$z(t) = D_2 g(t, x(t, s_0, x_0, u_0), u_0(t))\, z(t) + D_3 g(t, x(t, s_0, x_0, u_0), u_0)\, v(t) + \int_{-h}^{0} (d_s B(t, s))\, v(t + s). \qquad \ldots(3.3)$$

Also, for every $W \in L_\infty$, we have

$$D_u x(t_0, s_0, x_0, u_0,)\, w = \psi(t_0, s_0, x_0, u_0, w)$$

where the mapping $t \to \psi(t, s_0, x_0, u_0, w)$ is the unique absolutely continuous solution of (3.3) satisfying the initial condition $(s_0, 0)$ i.e. $z(s_0) = 0$.

PROOF : Let D_u be the partial derivative of $x(t, s_0, x_0, u)$ with respect to u. Then,

$$D_u x(s_0, s_0, x_0, u) = 0 \text{ in } [-h, 0]$$

$$D_u x(t, s_0, x_0, u) = \int_{s_0}^{t} D_2 g(s, x(s, s_0, x_0, u), u(s)).\, D_u x(s, s_0, x_0, u)\, ds + \int_{s_0}^{t} D_3 g(s, x(s, s_0, x_0, u), u(s))\, ds + \int_{s_0}^{t} \int_{-h}^{0} (d_s B(t, s))\, ds. \qquad \ldots(3.4)$$

Differentiating with respect to t, we obtain

$$\frac{d}{dt} D_u x(t, s_0, x_0, u)\, v = D_2 g(s, x(s, s_0, x_0, u), u).\, D_u x(s, s_0, x_0, u)\, v + D_3 g(s, x(s, s_0, x_0, u), v + \int_{-h}^{0} (d_s B(t, s))\, v. \qquad \ldots(3.5)$$

Substituting the limits of integration in (3.4), we obtain

$$D_u x(s_0, s_0, x_0, u)\, v = D_u x_0\, v = 0$$

proving the second assertion. To prove the first part, we note that

$$\bar{D} x(t, s_0, x_0, u)(h, v) = D_3 x(t, s_0, x_0, u) + D_4 x(t, s_0, x_0, u)\, v.$$

But,

$$D_3 x(t, s_0, x_0, u) = I + \int_{s_0}^{t} D_2 g(s, x(s, s_0, x_0, u), u(s)) \times D_3 x(s, s_0, x_0, u), u(s))\, ds$$

$$D_4\, x\,(t, s_0, x_0, u) = \int_{s_0}^{t} D_3\, g\,(s, x\,(s, s_0, x_0, u), u\,(s))\, ds$$

$$+ \int_{s_0}^{t} \left[\int_{-h}^{0} (d_s\, B\,(t, s))\right] ds.$$

Therefore,

$$\bar{D}x\,(t, s_0, x_0, u)\;\;(h, v) = h + \int_{s_0}^{t} D_2\, g\,(s, x\,(s), u\,(s)).\; D_3\, x\,(s, s_0, x_0, u)\; hds$$

$$+ \int_{s_0}^{t} D_3\, g\,(s, x\,(s, s_0, x_0, u), u\,(s))\; vds$$

$$+ \int_{s_0}^{t} \int_{-h}^{0} (d_s\, B\,(t, s))\; vds. \qquad \ldots(3.6)$$

Taking the t derivative, we obtain :

$$\frac{d}{dt}\, \bar{D}x\,(t, s_0, x_0, u)\,(h, v) = 0 + D_2\, g\,(s, x\;(s, s_0, x_0, u), u\,(s))$$

$$\times\; D_3\, x\,(S))\; h + D_3 g\;(s, x\,(s, s_0, x_0, u), u\,(s))\; v$$

$$+ \int_{-h}^{0} (d_s\, B\,(t, s))\; v.$$

Moreover, from (3.6), we have

$$\bar{D}x\,(s_0, s_0, x_0, u)\,(h, v) = h + 0 + 0 + 0.$$

Hence, the Proposition.

Definition 3.1— Let X, Y be Banach Manifolds of class C^1 and let $h : X \to Y$ be a C^1 mapping. A point y_0 in $h\,(X)$ is called a normal value of h if there exists at least one $x_0 \in h^{-1}\,(y_0)$ such that the differential dh_{x_0} is a split-surjective linear mapping.

The next Lemma gives a necessary and sufficient condition for the derivative $D_u\, x\,(t, s_0, x_0, u_0)$ to be a surjective linear mapping, in which case the point $x\,(t, s_0, x_0, u_0)$ is a normal value.

Lemma 3.2— Given $x_0 \in E^n$, $u \in L_\infty$. For $t_1 > s_0 + h$, let $t \in [s_0 \cdot\;\; t_1]$. Let $u \to x\,(t_1, s_0, x_0, u)$ be the mapping $T : L_\infty \to E^n$, defined by

$$T\,(u) = x\,(t_1, s_0,\; x_0, u)$$

where $x\,(t_1, s_0, x_0, u)$ is a solution of (3.2). Then $DT\,(u) = \dfrac{d}{du}\, T\,(u) : L_\infty \to E^n$ has a continuous local right inverse (is a surjective linear mapping) if and only if the linear

variational control system (3.3) of (3.2) along the response $t \rightarrow x(t_1, s_0, x_0, u)$ is controllable on $[s_0, t_1]$, $t_1 > s_0 + h$.

PROOF : For $t \in [s_0, t_1]$, $t_1 > s_0 + h$, $u \in L_\infty$, let $t \rightarrow z(t, t_0, x_0, u)$ denote the solution of (3.3). Clearly, the linear variational control system of (3.2) along the solution $t \rightarrow x(t, s_0, x_0, u)$ is controllable on $[s_0, t_1]$ if and only if the mapping $u \rightarrow z(t, s_0, x_0, u)$ is surjective, $t \in [s_0, t_1]$. But by Lemma 3.1,

$$z(t_1, s_0, x_0, u, v) = D_u x(t_1, s_0, x_0, u) = DT(u)\, v.$$

Therefore, the mapping $v \rightarrow z(t_1, s_0, x_0, u, v)$ is surjective if and only if $DT(u)$ is surjective. The following proposition will be made use of in the next theorem.

Proposition 3.1 (Surjective mapping theorem) (Graves)— Let U be open in a Banach space X. Let $f: U \rightarrow Y$ be a C^1 map into a Banach space Y. Let $x_0 \in X$. If $f'(x_0)$ is surjective, then f is locally open in a neighbourhood of x_0. More precisely, there exists an open neighbourhood V of x_0 contained in U having the following Lang[6] property :

For each $x \in V$, and open ball B_x centered at x, contained in V, the image $f(B_x)$ contains an open neighbourhood of $f(x)$.

Remarks 3.1: This is another form of the implicit function theorem given by (p. 193).

Theorem 3.1— The system (3.2) is controllable on $[s_0, t]$, $t > s_0 + h$, whenever the system

$$\dot{z}(t) = D_2 g(t, 0, 0)\, z(t) + D_3 g(t, 0, 0)\, v(t) + \int_{-h}^{0} (d_s B(t, s))\, v(t+s)\, ds \qquad \ldots(3.7)$$

is controllable on $[s_0, t]$, $t > s_0 + h$.

PROOF : For system (3.2), let $u \in L_\infty$, and $x(s_0, x_0, u)$ be its response. Let $T: L_\infty \rightarrow E^n$ be the map defined by $Tu = x(t, s_0, 0, u)$.

It follows from the conditions on g and (2.11) that $T(IU) = IR(t, s_0)$. Suppose system (3.7) is controllable, then by Lemmas 3.1 and 3.2,

$$DT(0) = \frac{d}{du}[T(u)]_{u=0} = D_4 x(t, s_0, 0, u) \mid u = 0$$

is a surjective linear mapping of $L_\infty \rightarrow E^n$. Therefore, by Lang[6] (p. 193), T is locally open. Hence, there is an open ball $B_p \subseteq L_\infty$ containing zero and an open ball $B_r \subseteq E^n$ containing zero such that $B_r \subseteq T(B_p)$.

Since IU contains an open ball containing zero, $r > 0$, $p > 0$ can be chosen such that $B_r \subseteq T(B_p \cap IU)$.

Therefore,

$$B_r \subseteq T(IU) = IR(t, s_0)$$

so that $0 \in \text{Int } IR(t, s_0)$.

Remarks 3.1 : The above theorem assures us that if the linear variational control system of (3.2) corresponding to the zero solution is controllable, then $0 \in \text{Int } IR(t, t_0)$.

Definition 3.2— The domain **D** of null controllability of (3.2) is the set of all initial points $x_0 \in E^n$, for which the solution of (3.2) with $x(t_0) = x_0$, satisfies $x(t_1) = 0 \in E^n$, at some t_1 using $U \in IU$.

Theorem 3.2— In system (3.2), assume

(i) $g(t, 0, 0) = 0, t > s_0$

(ii) system (3.7) is controllable on $[s_0, t]$, $t > s_0 + h$

and

(iii) the smoothness assumptions hold for g

then

the domain of null controllability of (3.2) contains zero in its interior.

PROOF : The variation of parameter formula for system (3.2) is given from (2.7) as

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, x(s), u(s))$$

$$\int_{-h}^{0} (d_s B(t, s))\, u(t + s)\, ds. \qquad \ldots(3.8)$$

We note that if there exists a $u \in IU$ such that the solution of (3.2) satisfies

$$x(t_0, t_0, x_0, u) = x_0,\ x(t_1, t_0, x_0, u) = 0,$$

then from (3.8) we have

$$0 = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, u, x(s)) \int_{-h}^{0} (d_s B(t, s))\, u(t + s)\, ds.$$

By (iii), (i) $0 \in \mathbf{D}$, the domain of null controllability of (3.2). Assume that $0 \notin \text{Int } \mathbf{D}$. Then there exists a sequence $\{x_i\}^{\infty}{}_{i=1} \subseteq E^n$ such that $x_i \to 0$ as $i \to \infty$ and no x_i is in **D**, hence, $x_i \neq 0$ for any i. Since $x_i \notin \mathbf{D}$ $x(t, t_0, x_i, u) \neq 0$, for any $t > t_0 + h$ and any $u \in IU$. Hence, from (3.8),

$$-G(t, t_0, x_i, u) \equiv \zeta_i \neq \int_{t_0}^{t} F(t, s, u, x(s)) \left[\int_{-h}^{0} (d_s B(t, s,))\, u(t + s) \right] ds$$

for any i, any $t > t_0 + h$ and $u \in IU$. Therefore, ζ_i is not contained in $IR(t, t_0)$ for any $t > t_0 + h$. Hence, we have a countable sequence $\{\zeta\}_i \subseteq E^n$, such that

$$\zeta_i \to 0 \text{ as } i \to \infty$$

$$\zeta_i \notin IR(t, t_0), \text{ for any } t > t_0 + h$$

$$\zeta_i \neq 0 \text{ for any } i$$

hence $0 \notin \text{Int } IR(t, t_0)$, for any $t > t_0 + h$.

But by (ii), the linear variational system of (3.2) about the zero solution is controllable, hence, by Theorem 3.1, $0 \in \text{Int } IR(t, t_0)$ $t > t_0 + h$ which is a contradiction. This contradiction shows that $0 \in \text{Int } \mathbf{D}$.

Remarks 3.3 : The above theorem says that, if the linear variational system is controllable, then (3.2) is locally null controllable with constraints.

Theorem 3.3— For system (3.2), assume

(i) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(ii) $g(t, 0, 0) = 0$;

(iii) system (3.7) is controllable on $[t_0, t_1]$, $t_1 > t_0 + h$;

(iv) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad \ldots(3.9)$$

is uniformly asymptotically stable so that every solution of (3.9) satisfies

$$\| x(t, t_0, x_0, 0) \| \leq k \mid e^{-\alpha(t-t_0)}, t > t_0, k > 0, \alpha > 0 \qquad \ldots(3.9a)$$

then

(3.2) is null controllable with constraints.

Proof : By (i), (ii), (iii) the domain $\mathbf{D}$ of null controllability of (3.2) contains zero in its interior. Therefore, there exists a ball $B_1 \in \text{Int } \mathbf{D}$.

By (iv) every solution of (3.9) which is a solution of (3.2) (with $u = 0$) satisfies $x(t, t_0, x_0, 0) \to 0$ as $t \to \infty$, hence, at some $t_1 < \infty$, we have $x(t_1, t_0, x_0, 0) \equiv x_2 \in B_1 \subseteq \text{Int } \mathbf{D}$. Therefore, with t_1, x_2 as initial data there exists $u \notin IU$ and some $t_2 > t_1$, such that the solution $x(t)$ satisfies $x(t_2, t_1, x_2, u) = 0$; proving the theorem.

We now give necessary and sufficient conditions for the controllability of the linear variational system (3.7).

Now, consider the linear system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + \int_{-h}^{0} (d_s H(t, s))\, u(t+s). \qquad \ldots(3.10)$$

Following Klamka[4], the unique solution of (3.10) is given by

$$x(t) = F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau)\, \left[\int_{-h}^{0} (d_s H(\tau, s))\, u(\tau + s) + B(\tau) \right] d\tau \qquad \ldots(3.11)$$

where $F(t, s)$ is the fundamental matrix for the homogeneous system

$$\dot{x}(t) = A(t)\, x(t). \qquad \ldots(3.12)$$

Following Klamka[4] and using the unsymmetric Fubini theorem, we can transform eqn. (3.11) to the following convenient form :

$$x(t) = F(t, t_0)\, x(t_0) + \int_{-h}^{0} d_H \left[\int_{t_0+s}^{t_0} F\, t\;\, l - s)\, H(l - s, s)\; u_{t_0}\, dl \right]$$

$$+ \int_{t_0}^{t} \left[\int_{-h}^{0} F(t, l - s)\, d\bar{H}(l - s, s) + F(t, l)\, B(l)\, u(l)\, dl \right]. \qquad \ldots(3.13)$$

We now define the $n \times n$ controllability matrix of (3.10) as

$$W_1(t_0, t_1) = \int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1, t-s)\, d\bar{H}(t-s, s) \right] \left[\int_{-h}^{0} F(t_1, t-s)\, dH(t-s, s) \right]^T dt$$

$$+ \int_{t_0}^{t_1} [F(t_1, t)\, B(t)]\, [F(t_1, t)\, B(t)]^T\, dt. \qquad \ldots(3.14)$$

The proof of the next theorem follows exactly the same proof given in Theorem 1 of Klamka[3] and therefore will be omitted.

Proposition 3.1— System (3.10) is controllable on $[t_0, t_1]$ if and only if

$$\text{rank } W_1(t_0, t_1) = n;\; t_1 > t_0. \qquad \ldots(3.15)$$

Remark 3.4 : Here, we extend and complement this result of Klamka[3] to system (1.1) in the following manner.

The next theorem is a consequence of Theorem 3.3 and Proposition 3.1. Let

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \ldots(3.16)$$

where

$$A(t) = D_2\, g(t, 0, 0),\; B(t) = D_3\, g(t, 0, 0) \qquad \ldots(3.17)$$

and let

$$W_1(t_0, t_1) = \int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1, t - s)\, d\bar{B}(t - s, s) \right] \left[\int_{-h}^{0} F(t, t-s)\, d\bar{B}(t-s, s) \right]^T$$

(equation continued on p. 226)

$$+ \int_{t_0}^{t_1} [F(t_1, t) B(t)] [F(t_1, t)]^T \, dt \qquad \ldots(3.18)$$

where $F(t, s)$ is the fundamental matrix for the homogeneous system

$$\dot{x}(t) = A(t) x(t). \qquad \ldots(3.18)$$

Theorem 3.4— For system (3.2), assume

(i) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(ii) $g(t, 0, 0) = 0$;

(iii) rank $W_1(t_0, t_1) = n$; $t_1 > t_0$

where $W_1(t_0, t_1)$ is given by (3.14);

and

(iv) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad (3.19)$$

is uniformly asymptotically stable then

(3.2) is null controllable with constraints.

PROOF : Immediate from Theorem 3.3 and Proposition 3.1.

4. APPLICATIONS

If we now specialize to the constant system with multiple delays in the control defined by

$$\dot{x}(t) = g(x(t), u(t)) + \sum_{i=1}^{k} C_i u(t - h_i) \qquad (4.1)$$

then the following results follow :

It is known[1] that the system

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{k} C_i u(t - h_i) \qquad \ldots(4.2)$$

is controllable on $[0, t_1]$, $t_1 > \max\{h_1, \ldots\ldots, h_m\}$ if and only if

$$\text{rank}\,[B, \ldots, A^{n-1} B, C_1, \ldots, A^{n-1} C_1 \ldots, C_k, \ldots A^{n-1} C_k] = \wedge. \qquad \ldots(4.3)$$

Remarks 4.1 : The above results of Sebakhy and Bayoummi[7] cannot be applied directly to the nonlinear system (4.1), without some modifications as will be shown by

Proposition 4.2. The linear variational system of (4.1) corresponding to the zero solution is given by

$$\dot{z}(t) = D_1 g(0, 0) z(t) + D_2 g(0, 0) u(t) + \sum_{i=1}^{k} C_i u(t - h_i). \quad \ldots(4.4)$$

If we set

$$A = D_1 g(0, 0);\ B = D_2 g(0, 0) \quad \ldots(4.5)$$

and suppose that A, B are autonomous, then the next Proposition follows from the reasoning above.

Proposition 4.1— The autonomous linear system (4.4) corresponding to the zero solution of (4.1) is controllable if and only if

$$\text{rank}\,[B, \ldots, A^{n-1} B, \ldots, C_k \ldots A^{n-1} C_k] = n$$

where A and B are given by (4.5).

The next Proposition follows from Theorem 3.4 and Proposition 4.1.

Proposition 4.2— For system (4.1), assume

(i) (4.2) is autonomous;

(ii) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(iii) $g(0, 0) = 0$;

(iv) rank $[B, \ldots, A^{n-1} B \ldots\ldots, C_k \ldots\ldots A^{n-1} C_k] = n$,

where A and B are defined by (4.5); and

(v) the system

$$\dot{x}(t) = g(x(t), 0) \quad \ldots(4.6)$$

is uniformly assymptotically stable then

(4.1) is null controllable with constraints.

PROOF : Immediate.

REFERENCES

1. J. P. Dauer, *J. Math. Anal. Appl.* 54 (1976) 717-25.
2. V. Q. Khanh, *Proceedings of the second International Conference held at Blazjewko, Poland*, May 3-10, 1981.

3. J. Klamka, *IEEE Trans. Automatic Control* **AC-21**, (1976), 594–95.
4. J. Klamka, *Int. J. Control* **28** (1978), 307–312.
5. J. Klamka, *IEEE Trans. Automatic Control* **31** (1980), 811–19.
6. S. Lang, *Analysis II.* Addison-Wesley Publishing Co., Reading M A. 1868.
7. O. Sebahkhy and M. M. Bayoummi, *Int. J. Control* **17** (1973), 127–35.
8. Y. Yamatoto, *J. Optim. Theory Appl.* **22** (1977), 41–49.

Indian J. pure appl. Math., 20(3) : 229–234, March 1989

A FINSLERIAN EXTENSION OF THE GRAVITATIONAL FIELD—II

SATOSHI IKEDA

Department of Mechanical Engineering, Faculty of Science and Technology Science University of Tokyo, Noda, Chiba 278, Japan

(*Received* 8 *March* 1988)

Some structural considerations are further made on the previously introduced Finslerian metrical structure $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y)$. In particular, some interesting features underlying the field equations for Finslerian gravitational field based on this metric are investigated.

1. INTRODUCTION

As has been mentioned in a previous paper[1], there appear two fields around each point x of the Finslerian gravitational field : One is the external (x)-field spanned by points $\{x\}$, which is nothing else than the Einstein's gravitational field[2], and the other is the internal (y)-field spanned by vectors $\{y\}$, which is regarded as the so-called internal space attached to x. The former is dominated by the four-dimensional Riemann (R_4) metric $\gamma_{\lambda\kappa}(x)$ $(\kappa, \lambda = 1, 2, 3, 4)$, while the latter is assumed, in general, to be governed by the four-dimensional Riemann (R_4) metric $h_{ij}(y)$ $(i, j = 1, 2, 3, 4)$.

From the vector bundle-like standpoint[4], the (y)-field may be regarded as a fibre at the point x of the base (x)-field and the total space of this vector bundle may be considered a unified field between the (x) and (y)-fields. The Finslerian gravitational field, therefore, may be likened to this unified field, presenting an aspect of eight-dimensional Riemannian structure (R_8) dominated by the R_8-metric $G_{AB}(X^A)$ $(X^A = (x^\kappa, y^i)$; $A = (\kappa, i) = 1, 2, 3, \ldots, 8)$ (Miron[3] and Miron and Anastasiei[4]). This R_8 structure has been reduced, in the previous paper[1], to a four-dimensional Finslerian structure (F_4) based on the Finsler metric $g_{\lambda\kappa}(x, y)$ (2.9) by means of a dimension reduction-process. These situations will be considered in more detail in the following.

2. ON THE FINSLERIAN STRUCTURE

In the previous paper (see section 3 of Ikeda[1]), we have taken account of inherent law internal vector y $(= y^i)$ in the form

$$\bar{y}^i = K^i_j(x)\, y^j \qquad \ldots(2.1)$$

where K^i_j represents the rotation matrix. From (2.1), one typical intrinsic parallelism (i.e., connection) of y such as

$$\delta y^i = dy^i + K^i_{j\mu}\, y^j\, dx^\mu \;(\equiv dy^i + N^i_\mu\, dx^\mu) = 0 \qquad \ldots(2.2)$$

has been proposed, where $K^i_{j\mu} \equiv -\dfrac{\partial K^i_j}{\partial x^\mu}$ and $dy^i \equiv \bar{y}^i - K^i_j(0)\, y^j$. The quantity N^i_μ in (2.2) plays the role of nonlinear connection in the theory of Finsler spaces[3,5] and also serves, in our case[1], as the mapping operator of the internal (y)-field on the external (x)-field (see below).

In our Finslerian field, which is regarded as the unified field mentioned above, the connection relations are prescribed by[3,4] [see (3.4) of Ikeda[1]]

$$DV^\kappa = dV^\kappa + F^\kappa_{\lambda\mu}\, V^\lambda\, dx^\mu + C^\kappa_{\lambda k}\, V^\lambda\, \delta y^k$$

$$DV^i = dV^i + F^i_{j\mu}\, V^j\, dx^\mu + C^i_{jk}\, V^j\, \delta y^k \qquad \ldots(2.3)$$

where

$$(F^\kappa_{\lambda\mu},\ F^i_{j\mu},\ C^\kappa_{\lambda k},\ C^i_{jk}) \qquad \ldots(2.4)$$

denote the coefficients of connection. From (2.3), the following covariant derivatives are introduced :

$$\left.\begin{aligned} V^\kappa{}_{|\mu} &= \frac{\delta V^\kappa}{\delta x^\mu} + F^\kappa_{\lambda\mu}\, V^\lambda, \quad V^\kappa|_k = \frac{\partial V^\kappa}{\partial y^k} + C^\kappa_{\lambda k}\, V^\lambda \\ V^i{}_{|\mu} &= \frac{\delta V^i}{\delta x^\mu} + F^i_{j\mu}\, V^j, \quad V^i|_k = \frac{\partial V^i}{\partial y^k} + C^i_{jk}\, V^j. \end{aligned}\right\} \qquad \ldots(2.5)$$

Therefore, the base and dual base (i.e., the adapted frames) are set by

$$\left.\begin{aligned} &\frac{\partial}{\partial \zeta^A} \equiv \left(\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^i_\mu \frac{\partial}{\partial y^i},\ \frac{\partial}{\partial y^i}\right) \\ &d\zeta^A \equiv (dx^\kappa,\ \delta y^i = dy^i + N^i_\mu\, dx^\mu). \end{aligned}\right\} \qquad \ldots(2.6)$$

From (2.6), the so-called decomposition factors are defined by

$$\left.\begin{aligned} &A^\kappa_A = (\delta^\kappa_\lambda,\ 0),\ A^A_\lambda = (\delta^\kappa_\lambda,\ -N^i_\lambda), \\ &B^i_A = (N^i_\lambda,\ \delta^i_j),\ B^A_i = (0,\ \delta^j_i) \end{aligned}\right\} \qquad \ldots(2.7)$$

by which the unified metric of the unified field G_{AB} is decomposed to, under the assumption that $G_{\lambda\kappa} = \gamma_{\lambda\kappa}(x)$, $G_{ij} = h_{ij}(y)$ and $G_{\lambda i} = G_{i\lambda} = 0$,

$$\left.\begin{aligned} g_{\lambda\kappa} &= A^A_\lambda \; A^B_\kappa \; G_{AB} = \gamma_{\lambda\kappa}(x) + N^i_\lambda \; N^j_\kappa \; h_{ij}(y) \\ g_{\lambda i} &= A^A_\lambda \; B^B_i \; G_{AB} = - \; N^j_\lambda \; h_{ji}(y) \\ g_{ij} &= B^A_i \; B^B_j \; G_{AB} = h_{ij}(y). \end{aligned}\right\} \qquad \ldots(2.8)$$

At this stage, as has been considered in Ikeda[1], if the (y)-field, is compactified, then only the F_4-metric $g_{\lambda\kappa}(x, y)$ $(2.8)_1$ appears above the surface. This compactification is likened geometrically to the mapping process of the (y)-field on the (x)-field, where the nonlinear connection N^i_λ plays the role of mapping operator. (In the previous paper[1], one new operator $e^i_\lambda\ (x)$, instead of $N^i_\lambda\ (x, y)$, has been introduced as the mapping operator). By this mapping process, those quantities such as δy^i, $F^i_{j\mu}$, $C^\kappa_{\lambda k}$, C^i_{jk}, etc. are brought to the external (x)-field and the unified field presents an aspect of F_4-structure based on

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y);$$

$$h_{\lambda\kappa}(x, y) \equiv N^i_\lambda \; N^j_\kappa \; h_{ij}(y). \qquad \ldots(2.9)$$

In the following, we shall pay our attention to some structural features underlying the field equations for this Finslerian gravitational field from the vector bundle-like standpoint based on (2.4).

3. On the Field Equations

Now, in our theory, the (y)-field itself is dominated by R_4-metric $h_{ij}(y)$, as mentioned in (2.8), so that its field equation may be written as

$$M_{ij}(y) - \tfrac{1}{2} M \; h_{ij}(y) = \mu_{ij}(y) \qquad \ldots(3.1)$$

where M_{ij} denotes the Ricci-tensor derived from the Riemannian curvature tensor of the (y)-field, M the scalar curvature and μ_{ij} means the energy-momentum tensor for this case. Then, (3.1) must be mapped on the external (x)-field governed by R_4-metric $\gamma_{\lambda\kappa}(x)$ by means of the mapping process mentioned above. As the result, therefore, the field equation of the unified is given by

$$R_{\lambda\kappa} - \tfrac{1}{2} R \, g_{\lambda\kappa} = \tau_{\lambda\kappa} \qquad \ldots(3.2)$$

where $R_{\lambda\kappa}$ is the Ricci-tensor derived from the third curvature tensor $R^\kappa_{\nu\lambda\mu}$ formed with $F^\kappa_{\lambda\mu}$, $R = R_{\kappa\lambda}\, g^{\kappa\lambda}$ and $\tau_{\lambda\kappa}$ denotes the energy-momentum tensor. In our, case, since $g_{\lambda\kappa}$ is given by (2.9), $R_{\lambda\kappa}$ has such a special form as

$$R_{\lambda\kappa}(F) = K_{\lambda\kappa}(\{\}) + M_{\lambda\kappa}(\Delta) \qquad \ldots(3.3)$$

where $K_{\lambda\kappa}$ represents the Riemannian Ricci-tensor, which is formed with the Christoffel three-index symbol $\left\{ {\kappa \atop \lambda\mu} \right\}$ with respect to $\gamma_{\lambda\kappa}(x)$, and $M_{\lambda\kappa}$ is defined as the rest, the latter being constructed by $\Delta^{\kappa}_{\lambda\mu}\left(\equiv F^{\kappa}_{\lambda\mu} - \left\{ {\kappa \atop \lambda\mu} \right\} \right)$. (This fact (3.3) is adapted to the definition $M_{\lambda\kappa} \equiv N^{i}_{\lambda} N^{j}_{\kappa} M_{ij}$). In (3.3), even if $R_{\lambda\kappa} = 0$, $M_{\lambda\kappa} \neq 0$ (i.e., $M_{ij} \neq 0$). This case corresponds to the compactification of the (y)-field considered by Gasperini[6].

Next, we shall take up the following field equation for the empty space :

$$S_{\nu\lambda} = 0 \qquad \ldots(3.4)$$

where $S_{\nu\lambda}$ means the Ricci-tensor derived from the first curvature tensor $S^{\kappa}_{\nu\lambda\mu}$ defined by $C^{\kappa}_{\lambda\mu}$ (cf. Miron and Anastasiei[4] and Matsumoto[5]). Concerning this, it has been shown[7] that if $S_{\nu\lambda} = 0$, then $S^{\kappa}_{\nu\lambda\mu} = 0$ holds good, so that the field itself presents an almost Riemannian aspect, due to Brickell's theorem[8]. That is to say, the Finslerian field with $S_{\nu\lambda} = 0$ becomes almost Riemannian. Therefore, (3.4) seems somewhat unsuitable from our Finslerian viewpoint. Further, following the definition of $S^{\kappa}_{\nu\lambda\mu}$, the Ricci-tensor $S_{\nu\lambda}$ ($\equiv S^{\kappa}_{\nu\lambda\kappa}$) in our case based on (2.9) is calculated as

$$S_{\nu\lambda} = \tfrac{1}{2}\,\gamma^{\kappa\alpha}\left(\frac{\partial^2 h_{\nu\lambda}}{\partial y^{\alpha}\,\partial y^{\kappa}} - \frac{\partial^2 h_{\kappa\lambda}}{\partial y^{\alpha}\,\partial y^{\nu}}\right) \qquad \ldots(3.5)$$

at least in the first order approximation with respect to $h_{\lambda\kappa}$. In (3.5), if $h_{\lambda\kappa} = \frac{1}{2}\,\dfrac{\partial^2 E(x, y)}{\partial y^{\lambda}\,\partial y^{\kappa}}$, then $S_{\nu\lambda} = 0$ (3.4) is identically satisfied. This is also unsuitable from a physical point of view[9].

The field equations (3.2) and (3.4) for the unified field may be reconsidered from the vector bundle-like standpoint as follows : In the total space, the adapted frame (2.6) has been set and the connection with the coefficients (2 4) has been introduced. This connection is made metrical for the metrical structure of the total space

$$G = G_{AB}\,d\zeta^{A}\,d\zeta^{B} = g_{\lambda\kappa}(x, y)\,dx^{\kappa}\,dx^{\lambda} + g_{ij}(x, y)\,\delta y^{i}\,\delta y^{j}. \qquad \ldots(3.6)$$

In this case, the compactification process with respect to (2.8) is not taken into account and the frame (2.6) is assumed to be suitably adapted to the conditions $g_{\lambda i} = g_{i\lambda} = 0$. By straightforward calculations, the following six kinds of curvature tensors in the total space are obtained through the Ricci-identities with respect to the convariant derivatives (2.5) :

$$\mathscr{R}^{A}_{BCD} = (R^{\kappa}_{\lambda\mu\nu}, R^{l}_{i\lambda\mu}, P^{l}_{j\lambda k}, P^{\kappa}_{\lambda\mu k}, S^{\kappa}_{\lambda ij}, S^{l}_{jkl}). \quad ...(3.7)$$

As to the Ricci-tensors, they are given by, from (3.7),

$$\mathscr{R}_{AB} (\equiv \mathscr{R}^{C}_{ABC}) \equiv (R_{\lambda\mu} \equiv R^{\kappa}_{\lambda\mu\kappa},\ P^{1}_{i\lambda} \equiv P^{\kappa}_{i\lambda k} - P^{2}_{\lambda i} \equiv P^{\kappa}_{\lambda k i},$$

$$S_{ij} \equiv S^{k}_{ijk}). \quad ...(3.8)$$

At this stage, following Mircon[4], we shall define the field equation for the total space in the form

$$\mathscr{R}_{AB} - \tfrac{1}{2} \mathscr{R} G_{AB} = \tau_{AB} \quad ...(3.9)$$

where the total scalar $\mathscr{R}$ $(\equiv \mathscr{R}_{AB} G^{AB})$ is given by $\mathscr{R} = R+S$ $(R = R_{\lambda\kappa} g^{\kappa\lambda}, S = S_{ij} g^{ij})$ and τ_{AB} means the eight-dimensional energy-momentum tensor depending on (x^{κ}, y^{i}). (3.9) is decomposed into the following four kinds of field equations[4] by (3.6) and (3.8):

$$\begin{aligned} &R_{\lambda\kappa} - \tfrac{1}{2} (R + S)\, g_{\lambda\kappa} = \tau_{\lambda\kappa} \\ &P^{1}{}_{i\lambda} = \tau_{i\lambda}, \\ &P^{2}{}_{\lambda i} = -\tau_{\lambda i} \\ &S_{ij} - \tfrac{1}{2} (S + R)\, g_{ij} = \tau_{ij}. \end{aligned} \quad ...(3.10)$$

These equations can be obtained quite systematically without any special conditions on the spatial structure itself Of course, they depend on the choice of adapted frame, so that different decompositions of (3.9) result in different equations. It turns out, therefore, that (3.2) and (3.4) are regarded as special cases of (3.10).

In (3.10), the most essential feature is the mixing of the $R_{\lambda\kappa}$-component representing the x-dependence and the S_{ij}-component representing the y-dependence. This means the interaction or coupling between the miro- and macro-degrees of freedom. The role of S in $(3.10)_1$ is compared to that of cosmological term, because S is considered an eight-dimensional effect (cf. Kerner[10]). It should be remarked that in the ordinary generalized Kaluza-Klein theory[6,10], such components as $P^{1}{}_{i\lambda}$ and $P^{2}{}_{\lambda i}$ (i.e., $\tau_{i\lambda}$ and $\tau_{\lambda i}$) are not taken into account from the beginning. It may be considered that (3.10) contains some more instructive information, so that its physical aspects should be investigated in future.

References

1. S. Ikeda, *Indian J. pure Appl Math.* **17** (1986), 813.
2. A. Einstein, *Ann. der. Phys.* **49** (1916), 769.
3. R. Miron, *Acta Sci. Math.* **49** (1985), 119.
4. R. Miron and M. Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to the Theory of Relativity* (Ed. : Acad. R. S. Romania) Bucuresti, 1987 (in Romanian).

5. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces.* Kaiseisha Press, Ohtsu, 1986.
6. M. Gasperini, *Nuovo Cimento*, **88 B** (1985), 172.
7. M. Matsumoto, *Rep. Math. Phys.* **8** (1975),, 103.
8. F. Brickel, *J. London Math. Soc.* **42** (1962), 325.
9. S. Ikeda and S. Dragomir, *Tensor N. S.* **44** (1987), 157.
10. R. Kerner, in *Differential Geometric Methods in Mathematical Physics* (ed. by P. L. Garcia and A. Pérez-Rendon). Springer, Berlin, 1987, pp. 150–73.

Indian J. pure appl. Math., 20 (3) : 235-241, March 1989

PERIODIC SOLUTIONS OF A CERTAIN FOURTH ORDER DIFFERENTIAL EQUATION

AYDIN TIRYAKI

Department of Mathematics, University of Erciyes, Kayseri 38039, *Turkey*

(*Received* 10 *March* 1988)

In this paper, we give sufficient conditions for the existence of periodic solutions of the nonautonomous equation (1.5).

1. INTRODUCTION

Consider the fourth-order constant coefficient differential equation :

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0. \qquad \text{.. (1.1)}$$

As it was shown in Ezeilo[2] that if

$$a_4 > \tfrac{1}{4} a_2^2 \qquad \text{...(1.2)}$$

or

$$a_1 a_3 < 0, a_4 \neq 0 \qquad \text{...(1.3)}$$

then the auxiliary equation corresponding to (1.1) has no purely imaginary roots whatever. By the general theory; this, in turn, implies first of all that (1.1) has no periodic solution whatever other than $x = 0$, and secondly that the perturbed equation

$$x^{(4)} + a_1 \ddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = p(t) \qquad \text{...(1.4)}$$

in which p ($\neq 0$) is any continuous w-periodic function of t, has an w-periodic solution subject to (1.2) or (1.3).

In the literature, there are some extensions of (1.2) or (1.3) in one form or other of the existence result for eqns. (1.4) where some of a_1, ... a_4 are not constants[1,2,5]. The object of the present paper is to extend the result (1.2) for (1.4) to equation in which a_1, a_2, a_3 and a_4 are not all constants.

We shall be concerned with the equation

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x})\ddot{x} + f_3(x) + f_4(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \qquad \text{.. (1.5)}$$

Where f_1, f_2, f_3, p are continuous functions depending only on the arguments shown and p is also assumed to be w-periodic in t. That is $p(t, x, y, z, u) = p(t + w, x,$

$y, z, u)$ for some $w > 0$ and for arbitrary t, x, y, z, u. We shall however require here that $f_4'(x)$ exists and is continuous for all x.

We shall establish here the following theorem :

Theorem—Suppose that

(i) there exists a constant $a_2 \geqslant 0$ such that

$$| f_2(y) | \leqslant a_2 \text{ for all } y \qquad \ldots(1.6)$$

$$a_4 \equiv \inf_x f_4'(x) > \tfrac{1}{4} a_2^2 \qquad \ldots(1.7)$$

(ii) there are constants $A_0 > 0$, $A_1 \geqslant 0$ such that

$$| p(t, x, y, z, u) | \leqslant A_0 + A_1 (| y | + | z |) \qquad \ldots(1.8)$$

for all t, x, y, z and u.

Then there exists a constant $\epsilon_0 > 0$ such that (1.5) has at least one w-periodic solution for all arbitrary f_1 and f_3 if $A_1 \leqslant \epsilon_0$.

It should be noticed that we get Theorem 3 given in Tejumola[4] under weaker conditions if we take $f_2(\dot{x}) = a_2$ and $f_4(x) = a_4 x$ in eqn. (1.5).

2. Preliminaries

As in Ezeilo[2], the proof will be by the Leray-Schauder technique, with the equation (1.5) embedded in the parameter-dependent equation :

$$x^{(4)} + \mu f_1(\ddot{x}) \dddot{x} + \{(1-\mu) a_2 + \mu f_2(\dot{x})\} \ddot{x} + \mu f_3(\dot{x}) + (1-\mu) a_4 x$$
$$+ \mu f_4(x) = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \; 0 \leqslant \mu \leqslant 1. \qquad \ldots(2.1)$$

Note that when $\mu = 1$ (2.1) reduces to the original eqn. (1.5). Also when $\mu = 0$ it reduces to the linear equation

$$x^{(4)} + a_2 \ddot{x} + a_4 x = 0$$

which, in view of the condition (1.2), has no non-trivial w-periodic solution. Thus Theorem will follow from the usual fixed point considerations in Ressig *et al.*[3] if it can be shown that there is a constant D whose magnitude is independent of μ $(0 \leqslant \mu \leqslant 1)$ such that, if $x(t)$ is any w-perodic solution of (2.1), then

$$| x(t) | \leqslant D, \; | \dot{x}(t) | \leqslant D, \; | \ddot{x}(t) | \leqslant D, \; | \dddot{x}(t) | \leqslant D \qquad \ldots(2.2)$$

for all $t \in [0, w]$. Note that the t-range here may be replaced by $[T, T + w]$ (arbitrary T) since we are dealing with a w-periodic $x(t)$.

Before proceeding to the actual verification of (2.2) we shall introduce some notations. Throughout what follows, D's with or without subscripts denote finite posi-

tive constants whose magnitudes depend on a_2 a_4, A_0, f_1 and f_3. The D's are all independent of μ. Finally a D without a subscript is not necessarily the same each time it occurs, but the numbered D's: D_0, D_1, ... retain a fiixed identity throughout.

3. Proof of Theorem

We shall take (2.1) in the more compact from

$$x^{(4)} + \mu\, f_1\,(\ddot{x})\,\dddot{x} + f_{2,\mu}\,(\dot{x})\,\ddot{x} + \mu f_3\,(\dot{x}) + f_{4,\mu}\,\mu\,(x) = \mu p\,(t,\, x,\, \dot{x},\, \ddot{x},\, \dddot{x})$$

$$(0 \leqslant \mu \leqslant 1) \qquad \text{...(3.1)}$$

by setting

$$f_{2,\mu}\,(\dot{x}) = (1 - \mu)\, a_2 + \mu f_2\,(\dot{x})$$

$$f_{4,\mu}\,(x) = (1 - \mu)\, a_4\,(x + \mu\, f_4\,(x).$$

In what follows in the rest of this paper $x\,(t)$ is an arbitrary w-periodic solution of (3.1). It will now be shown that $x\,(t)$ satisfies (2.2) if A_1 is sufficiently small.

Our main tool in the verification of (2.2) for $x\,(t)$ is the function,

$$V\,(t) = \mu \int_0^{\ddot{x}} z\, f_1\,(z)\;dz + \ddot{x}\int_0^{\dot{x}} f_{2,\mu}\,(y)\,dy + \dddot{x}\,\ddot{x} + \dot{x}\, f_{4,\mu}\,(x) + \mu \int_0^{\dot{x}} f_3\,(y)\,dy.$$

An elementary differentiation gives

$$\dot{V}\,(t) = U_0 + \mu\,\ddot{x}\,p\,(t,\, x,\, \dot{x}\,\ddot{x},\, \dddot{x}) \qquad \text{...(3.2)}$$

where

$$U_0 = \dddot{x}^2 + \dddot{x}\int_0^{\dot{x}} f_{2,\mu}\,(y)\,dy + \dot{x}^2 f'_{4,\mu}\,(x)$$

$$\geqslant \dddot{x}^2 - a_2\,|\,\dddot{x}\,|\;|\,\dot{x}\,| + a_4\,\dot{x}^2 \qquad \text{...(3.3)}$$

by (1.6) and (1.7).

It is now not difficult to verify that

$$U_0 \geqslant D_0\,(\dddot{x}^2 + \dot{x}^2) \qquad \text{...(3.4)}$$

for sufficiently small D_0. For, suppose for example that $D_0 < 1$, then, by (3.3),

$$U_0 - D_0\;(\dddot{x}^2 + \dot{x}^2) \geqslant (1 - D_0)\,\dddot{x}^2 - a_2\,|\,\dddot{x}\,|\;|\,\dot{x}\,| + (a_4 - D_0)\,\dot{x}^2$$

$$\geqslant \tfrac{1}{4}\,(1 - D_0)^{-1}\,U_1\,\dot{x}^2$$

where

$$U_1 = (4a_4 - a_2^2) - 4D_0\,(1 + a_4) + 4D_0^2.$$

However, by (1.7), $4a_4 - a_2^2 > 0$ and so U_0 is strictly positive if, say

$$0 < D_0 < \tfrac{1}{8}\,(4a_4 - a_2^2)\,(1 + a_4)^{-1}.$$

Thus, $U_0 - D_0\,(\dddot{x}^2 + \dot{x}^2) \geqslant 0$ if D_0 is sufficiently small, which gives (3.4) and hence by (3.2) and (1.8), leads to the estimate :

$$\begin{aligned}\dot{V}(t) &\geqslant D_0\,(\dddot{x}^2 + \dot{x}^2) - \{A_0\,|\,\ddot{x}\,|\,|\,\dot{x}\,| + A_1\,\ddot{x}^2\} \\ &\geqslant D_0\,\dddot{x}^2 + (D_0 - A_1/2)\,\dot{x}^2 - A_0\,|\,\ddot{x}\,| - \tfrac{3}{2}\,A_1\,\ddot{x}^2 \qquad \ldots(3.5) \\ &\geqslant D_1\,(\dddot{x}^2 + \dot{x}^2) - \tfrac{8}{5}\,A_1\,\ddot{x}^2 - D_2\end{aligned}$$

for some D_1, D_2 if A_1 is taken sufficiently small.

Because of the (assumed) w-periodicity of $x\,(t)$, we have, on integrating (3.5), that

$$0 \geqslant D_1 \int_0^w (\dddot{x}^2 + \dot{x}^2)\,dt - \tfrac{8}{5}\,A_1 \int_0^w \ddot{x}^2\,dt - D_2\,w. \qquad \ldots(3.6)$$

Combined with the inequality

$$\int_0^w \ddot{x}^2\,dt \leqslant \tfrac{1}{4}\,w^2\,\pi^{-2} \int_0^w \dddot{x}^2\,dt \qquad \ldots(3.7)$$

which can be verified by substituting Fourier expansions of $\ddot{x}$ and $\dddot{x}$ in (3.6), (3.7) leads to the estimate

$$(D_1 - \tfrac{2}{5}\,w^2\,\pi^{-2}\,A_1) \int_0^w \dddot{x}^2\,dt + D_1 \int_0^w \dot{x}^2\,dt \leqslant D_2\,w.$$

Therefore, if A_1 is further fixed such that

$$A_1\,w^2\,\pi^{-2} \leqslant \tfrac{5}{4}\,D_1$$

as we assume henceforth, then

$$D_1 \int_0^w (\dddot{x}^2 + \dot{x}^2)\,dt \leqslant 2D_2\,w. \qquad \ldots(3.8)$$

In particular

$$\int_0^w \dddot{x}^2\,dt \leqslant D_3.$$

Considering now identity

$$\ddot{x}(t) = \ddot{x}(T_1) + \int_{T_1}^{t} \dddot{x}^2(s)\, ds$$

with T_1 fixed (as is possible in view of the periodicity condition $\dot{x}(0) = \dot{x}(w)$) such that $\ddot{x}(T_1) = 0$, we have that

$$\max_{0<t\leqslant w} |\ddot{x}(t)| \leqslant \int_0^w |\ddot{x}(s)|\, ds \leqslant w^{1/2} \left(\int_0^w \dddot{x}^2(s)\, ds\right)^{1/2}$$

by Schwarz's inequality. Thus (3.8) implies that

$$\max_{0<t<w} |\ddot{x}(t)| \leqslant D_3^{1/2}\, w. \qquad \ldots(3.9)$$

From this, on referring to the identity

$$\dot{x}(t) = \dot{x}(T_2) + \int_{T_2}^{t} \ddot{x}(s)\, ds.$$

With T_2 chosen such that $\dot{x}(T_2) = 0$ (the choice being possible in view of the periodicity condition $x(0) = x(w)$), we have that

$$\max_{0<t\leqslant w} |\dot{x}(t)| \leqslant D_3^{1/2}\, w^2. \qquad \ldots(3.10)$$

To obtain an estimate for $|x(t)|$ first note that, because of the w-periodicity of $x(t)$, integration of both sides of (3.1) yields the result

$$\int_0^w -\{f_{4,\mu}(x) - \mu p(t, x, \dot{x}, \ddot{x}\, \dddot{x})\}\, dt = \mu \int_0^w f_3(\dot{x})\, dt$$

or indeed, in view of (3.10), that

$$\left| \int_0^w \{f_{4,\mu}(x) - \mu p(t, x\, \dot{x}, \ddot{x}, \dddot{x})\}\, dt \right| \leqslant D_4 \qquad \ldots(3.11)$$

by (1.8), (3.9) and (3.10)

$$|\mu p(t, x, \dot{x}, \ddot{x}, \dddot{x})| \leqslant D_3^{1/2}\, w$$

for some D_3. Also, since $a_4 > 0$ and $f_4(x) \operatorname{sgn} x \to \infty$ as $|x| \to \infty$ (by (1.7)), there clearly exists D_5 independent of μ such that

$$|f_{4,\mu}(x) - \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x})| \geqslant 2D_4\, w^{-1} \qquad \ldots(3.12)$$

if $|x(t)| \geqslant D_5$ for all $t \in [0, w]$. It is thus clear that $|x(T_3)| \leqslant D_5$ for some T_3, as otherwise, by (3.12), the left handside of (3.11) would be not less in magnitude than $2D_4$. The result that $|x(T_3)| \leqslant D_5$ combined with (3.10) to yield the required boundedness estimate for x:

$$\max_{0<t\leqslant w} |x(t) \leqslant D_4 + D_3^{1/2} w^3. \qquad \ldots(3.13)$$

It remains now to obtain estimate for $|\dddot{x}(t)|$ in order to complete our verification of (2.2). Multiplying (3.1) by $x^{(4)}$ and integrating from $t = 0$ to $t = w$ we obtain

$$\int_0^w (x^{(4)})^2\, dt = -\mu \int_0^w f_1(\ddot{x})\, \dddot{x}\, x^{(4)}\, dt + \int_0^w x^{(4)}\, Q dt \qquad \ldots(3.14)$$

where

$$Q = \mu p(t, x\ \dot{x}, \ddot{x}, \dddot{x}) - \{f_{2,\mu}(\dot{x})\,\ddot{x} - \mu f_3(\dot{x}) - f_{4,\mu}(x)\}$$

satisfies

$$|Q| \leqslant D_6 \qquad \ldots(3.15)$$

by (1.6), (1.8), (3.9), (3.10) and (3.13). But by (3.9), $|f_1(\ddot{x})| \leqslant D_7$, so that

$$|-\mu \int_0^w f_1(\ddot{x})\, \dddot{x}\, x^{(4)}\, dt| \leqslant D_7 (\int_0^w (x^{(4)})^2\, dt)^{1/2}$$

by (3.8). Thus from (3.14) and (3.15), we have that

$$\int_0^w (x^{(4)})^2\, dt \leqslant D_8 (\int_0^w (x^{(4)2}\, dt)^{1/2}$$

and hence that

$$\int_0^w (x^{(4)})^2\, dt \leqslant D_9$$

for some D_9. The result

$$\max_{0<t\leqslant w} |\ddot{x}(t)| \leqslant w^{1/2} D_9^{1/2} \qquad \ldots(3.16)$$

now follows readily.

The result (3.9), (3.10), (3.13) and (3.16) fully verify (2.2) for the arbitrary chosen w-periodic solution $x(t)$ of (3.1) if the A_1 in (1.8) is sufficiently small. This now completes the proof of the Theorem.

REFERENCES

1. J. O. C. Ezeilo, *Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mat e nat., ser VIII*, 63 (1977), 204–11.
2. J O. C. Ezeilo, *Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mate nat.*, 66 (1979), 344–50.
3. R. Reiss g, G. Sansone and R. Conti, *Nonlinear Differential Equations of Higher Order*, Leyden. Noordhoff International Publishing Co. 1974, p.p. 95-98.
4. H. O. Tejumola, *Ann. Mat. Pura Appl.*, **80** (1968), 177-96.
5. H. O. Tejumola, Existence of periodicso lutions for certain nonlinear fourth order differential equations, Paper presented at the Symposium on "Contributions to the development of Mathematics in Nigeria in honour of J. O. C. Ezeilo"

Indian J. pure appl. Math., 20 (3) : 242-249, March 1989

CUBIC TRANSFORMATIONS OF FINSLER SPACES AND n FUNDAMENTAL FORMS OF THEIR HYPERSURFACES

B. N. PRASAD

Department of Mathematics, St. Andrew's College, Gorakhpur

AND

J. N. SINGH

Department of Mathematics, Kisan P. G. College, Bahraich

(*Received* 10 *August* 1987; *after revision* 16 *August* 1988)

If $L(x, y)$ is the metric function of a Finsler space $F^n = (M^n, L)$ and β is a one-form $b_i(x)\, y^i$ in F^n, then the trannsformation $L \to L^* = f(L, \beta)$ known as β-change has been introduced by Shibata[9]. In the present paper we consider a particular case of β changes known as cubic transformation given by $L \to L^* = (L^3 + \beta^3)^{1/3}$. The relation between n fundamental forms of tangent Riemannian hypersurface of (M^n, L) and (M^n, L^*) has been obtained.

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n dimensional Finsler space with fundamental metric function $L(x, y)$. In general $L(x, y)$ is a function of point $x\,(= x^i)$ and element of support $y\,(= y^i)$ and positively homogeneous of degree one in y. Let (M^n, L') and (M^n, L'') be Finsler spaces whose metric functions $L'(x, y)$ and $L''(x, y)$ are obtained from L by the relations

$$L' = L + \beta \qquad \text{...(1.1)}$$

$$L''^2 = L^2 + \beta^2 \qquad \text{...(1.2)}$$

where $\beta = b_i\, y^i$, $b_i(x)$ is a component of a covariant vector which is a function of position alone. These two transformations have been introduced by Matsumoto[1] which have the geometrical properties stated therein. Generalizing these transformations, Chibata[9] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from L by the relation $L^*(x, y) = f(L, \beta)$, where f is positively homogeneous of degree one in L and β. This change of metric function is called a β-change. A particular case of β-changes is a cubic transformation of metric function given by

$$L^{*3} = L^3 + \beta^3. \qquad \text{...(1.3)}$$

The n fundamental forms of a Riemannian hypersurface of a Riemannian space have been defined and their properties have been studied by Rund[7,8]. Prasad[5] has obtained the relation in n fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L') whose metric functions are related by (1.1). He[6] also obtained the relation in n fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L'') whose metric functions are related by (1.2). In this paper we shall obtain the relation in n fundamental forms of tangent Riemannian hypersurfaces of $F^n = (M^n, L)$ and $F^{*n} = (M^n, L^*)$ whose metric functions are related by (1.3).

2. The Relation Between v-Curvature Tensors of F^n and F^{*n}

Throughout this paper the quantities corresponding to the Finsler space F^{*n} will be written by putting '*'. For the cubic transformation (1.3) the relation between the angular metric tensors h_{ij} and h^*_{ij} of F^n and F^{*n} will be given by

$$h^*_{ij} = p\, h_{ij} + 2pq\, m_i\, m_j \qquad \ldots(2.1)$$

where $p = L\, L^{*-1}$, $q = \beta\, L^{*-1}$, $m_i = q\, l_i - pb_i$ and $l_i = \partial_i L$. Thus the metric tensor g^*_{ij} of F^{*n} and its recipocal g^{*ij} are given by

$$\begin{aligned} g^*_{ij} &= pg_{ij} + pq^3\, l_i\, l_j - p^2\, q^2\, (l_i\, b_j + l_j\, b_i) \\ &\quad + q\,(l + p^3)\, b_i\, b_j \end{aligned} \qquad \ldots(2.2)$$

$$\begin{aligned} g^{*ij} &= p^{-1}\, g^{ij} - pq^3\, \lambda\, (p + qb^2)\, l^i\, l^j + q^2\, \lambda\, (l^i\, b^j + l^j\, b^i) \\ &\quad - 2pq\, \lambda\, b^i\, b^j \end{aligned} \qquad \ldots(2.3)$$

where we put

$$b^i = g^{ij}\, b_j,\ l^i = g^{ij}\, l_j,\ b^2 = g^{ij}\, b_i\, b_j,$$

$$\lambda^{-1} = p^3 - q^3 + 2p^2\, qb^2.$$

The differentiation of (2.2) with respect to y^k gives the (h) hv-torsion tensor $C^*_{ijk} = (\partial\, g^*_{ij}/\partial y^k)/2$ of F^{*n}:

$$\begin{aligned} C^*_{ijk} &= pC_{ijk} + pq^2\,(h_{ij}\, m_k + h_{ik}\, m_j + h_{jk}\, m_i)/2L \\ &\quad - (2p^3 - 1)\,(p/L)\, m_i\, m_j\, m_k. \end{aligned} \qquad \ldots(2.4)$$

Paying attention to $m_i\, y^i = 0$, $h_{ij}\, y^i = 0$, $C_{ijk}\, y^i = 0$, we obtain

$$C^{*h}_{ij} = C^*_{ijk}\, g^{*hk} = C^h_{ij} + (q^2/2L)\,(h^h_i\, m_j + h^h_j\, m_i)$$

(equation continued on p. 244)

$$+ pq\lambda (ql^h - 2pb^h) C_{ij} + p^2 q^3 (p + qb^2)(\lambda/2L) h_{ij} l^h$$
$$- (pq^2 \lambda/2L) h_{ij} b^h - (q\lambda/L)[p^2(2p^3 - l)(p + qb^2)$$
$$+ q^3] m_i m_j l^h + (p\lambda/L) m_i m_j b^h \qquad ...(2.5)$$

where

$$C_{ij} = C_{ijk} b^k.$$

We shall now find the v-curvature tensor S^*_{hijk} which is given by $S^*_{hijk} = C^{*m}_{ij}$ $C^*_{hkm} - C^{*m}_{ik} C^*_{hjm}$. By the use of (2.4) and (2.5) we get

$$S^*_{hijk} = p\, S_{hijk} + C_{ij} d_{hk} + C_{hk} d_{ij} - C_{ik} d_{hj} - C_{hj} d_{ik}$$
$$+ h_{ij} E_{hk} + h_{hk} E_{ij} - h_{ik} E_{hj} - h_{hj} E_{ik} \qquad ...(2.6)$$

where

$$d_{hk} = (p^2 \lambda/L) m_h m_k - pq^3 C_{hk} - (p^2 q^2 \lambda/4L) h_{hk} \qquad ...(2.7)$$
$$E_{hk} = (pq^2\lambda/4L^2)[q^2(2p^3 + 1) - 2p^5 b^2] m_h m_k$$
$$- (pq^4 \lambda/8L^2)(q^2 - p^2 b^2) h_{hk} - (p^2 q^2 \lambda/4L) C_{hk}. \qquad ...(2.8)$$

Remark : It is to be noted that if $L(x, y)$ is the metric function of a Riemannian space i.e. $L(x, y) = ((a_{ij}(x) y^i y^j)^{1/2} = \alpha$ then $L^*(x, y) = (\alpha^3 + \beta^3)^{1/3}$. Thus F^{*n} is a Finsler space with an (α, β) metric. It is known that a Finsler space with an α, β-metric is semi-C-reducible[4]. In order to clearify this fact in this case, calculating from (2.4) we have

$$C^*_{ijk} = U(h^*_{ij} C^*_k + h^*_{jk} C^*_i + h^*_{ik} C^*_j)/(n + 1)$$
$$+ VC^*_i C^*_j C^*_k / C^{*2} \qquad ...(2.9)$$

where

$$U = (n + 1) q^2 A/2L,\; V = -(1 + q^3) C^{*2} A^3/L^*,\; C^{*2} = g^{*ij} C^*_i C^*_j,$$

and

$$A = 2L/\{(n + 3\lambda) q^2 - 2p^2 b^2 \lambda\}.$$

3. Hypersurface of (M^n, L)

Let (M^{n-1}, L) be a hypersurface of (M^n, L) given by the equation

$$x^i = x^i(u^\alpha). \qquad ...(3.1)$$

Let us suppose that the functions (3.1) are at least of class C^3 in u^α and the projection

factors $B^j_\alpha = \partial x^j/\partial u^\alpha$ are such that their matrix has maximal rank $n-1$. The fundamental metric function $\mathbf{L}(u, v)$ of the hypersurface is given by

$$L(u^\alpha, v^\alpha) = L(x^i(u^\alpha),\ B^i_\alpha\ v^\alpha)$$

where v^α is the element of support for the hypersurface for which

$$y^i = = B^j_\alpha\ v^\alpha. \qquad \text{...(3.2)}$$

Thus if l^α denote the normalized vector along the element of support then $l^i = B^i_\alpha\ l^\alpha$. If $g_{hj}(x, y)$ denote the metric tensor of (M^n, L), the induced metric tensor of $(M^{n-1}, \mathbf{L})$ is given by

$$g_{\alpha\beta}(u, v) = g_{hj}(x, y)\ B^h_\alpha\ B^j_\beta\,. \qquad \text{...(3.3)}$$

The inverse of (3.3) is denoted by $g^{\alpha\beta}(u, v)$ by means of which we define the quantities

$$B^\alpha_i\ (u, v) = g^{\alpha\beta}(u, v)\, g_{ij}(x, y)\ B^j_\beta\,. \qquad \text{...(3.4)}$$

The unit normal vector $N^j(u, v)$ of $(M^{n-1}, \mathbf{L})$ is determined by the relations

$$g_{hj}(x, y)\, B^h_\beta\ N^j(u, v) = 0,\ g_{hj}(x, y)\, N^h(u, v)\, N^j(u, v) = 1. \qquad \text{...(3.5)}$$

We have the following identities from (3.3), (3.4) and (3.5) :

$$B^\alpha_j\ B^j_\beta = \delta^\alpha_\beta\,,\ B^j_\alpha\ B^\alpha_h + N^j N_h = \delta^j_h \qquad \text{...(3.6)}$$

where

$$N_i = g_{ij}(x, y)\, N^j.$$

If $C_{hjk}(x, y)$ denotes the (h) hv-torsion tensor of $(M^n\ L)$, the induced (h) hv-torsion tensor $C_{\alpha\beta\gamma}(u, v)$ of $(M^{n-1}, \mathbf{L})$ is given by

$$C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y)\ B^h_\alpha\ B^j_\beta\ B^k_\gamma \qquad \text{...(3.7)}$$

from which we obtain

$$C^\alpha_{\beta\gamma} = B^\alpha_i\ C^i_{jk}\ B^j_\beta\ B^k_\gamma\,. \qquad \text{...(3.8)}$$

The relative v-covariant derivative of the projection factor B^i_α with respect to the induced Cartan connection $IC\ \Gamma$ is given by[2]

$$B^i_{\beta|\gamma} = - B^i_\alpha C^\alpha_{\beta\gamma} + C^i_{hk} B^h_\beta B^k_\gamma. \qquad ...(3.9)$$

This tensor is normal to (M^{n-1}, L). Therefore we may write

$$B^i_{\beta|\gamma} = M_{\beta\gamma} N^i. \qquad ...(3.10)$$

From (3.9) it is clear that $M_{\beta\gamma}$ is symmetric in β and γ and it may be written as

$$M_{\beta\gamma} = C_{ijk} N^i B^j_\beta B^k_\gamma. \qquad ...(3.11)$$

The tangent vector space M^{n-1}_x to M^{n-1} at every point $x^i (= u^\alpha)$ of the hypersurface is considered as the Riemannian space (M^{n-1}_x, g_x) with the Riemannian metric $g_x = g_{\alpha\beta}(u, v)\, dv^\alpha\, dv^\beta$. The components of the (h) hv-torsion tensor $C^\alpha_{\beta\gamma}$ will be the Christoffel symbols associated with g_x. If M^n_x is the tangent vector space to M^n at $x^i (= u^\alpha)$, then (M^{n-1}_x, g_x) will be the hypersurface of (M^n_x, g_x) given by (3.2), where $g_x = g_{ij}(x, y)\, dy^i\, dy^j$ is the Riemannian metric on M^n_x. The quantities $M_{\beta\gamma}$ given by (3.11) will be considered as the coefficients of the second fundamental form of the tangent Riemannian space (M^{n-1}_x, g_x).

In general the coefficients of the rth fundamental form of (M^{n-1}, g_x) are defined as[7]

$$C_{(1)\alpha\beta} = g_{\alpha\beta},\ C_{(2)\alpha\beta} = M_{\alpha\beta},\ C_{(r)\alpha\beta} = C_{(r-1)\alpha\delta}\, M^\delta_\beta\ (2 \leqslant r \leqslant n) \qquad ...(3.12)$$

where

$$M^\delta_\beta = g^{\alpha\delta} M_{\alpha\beta}.$$

4. The n Fundamental Forms of a Hypersurface of (M^n, L^*)

Let (M^{n-1}, L^*) be a hypersurface of $(M^n\ L^*)$ given by the same equation (3.1). It is to be noted that a unit normal vector N^i to (M^{n-1}, L^*) is not necessarily normal to (M^{n-1}, L^*). Paying attention to $l_i N^i = 0$, from (2.2) we have

$$g^*_{ij} B^i_\alpha N^j = (b_i N^i) \{q(1 + p^3) b_\alpha - p^2 q^2 l_\alpha\} \qquad ..(4.1)$$

$$g^*_{ij} N^i N^j = p + q(1 + p^3)(b_i N^i)^2 \qquad ...(4.2)$$

where $b_\alpha = b_l B^l_\alpha$. If b_l is tangential to the hypersurface (M^{n-1}, L), that is, $b_l N^l = 0$, then we have

$$N^{*i} = p^{-1/2} N^i \qquad ...(4.2)$$

where we have chosen a normal vector N^{*i} to (M^{n-1}, L^*) in the same direction as N^i. Hence we have the following :

Theorem 4.1—Let (M^n, L^*) be a Finsler space obtained from a Finsler space (M^n, L) by the transformation (1.3). If (M^{n-1}, L^*) and (M^{n-1}, L) are the hypersurface of these spaces given by the same equation (3.1) and b_l is tangential to the hypersurface (M^{n-1}, L), then the vector normal to (M^{n-1}, L) is also normal to (M^{n-1}, L)

Now we establish the following :

Theorem 4.2—Let (M^n, L^*) be a Finsiler space obtained from a Finsler space (M^n, L^*) by the transformation (1.3). Let (M^{n-1}, L^*) and (M^{n-1}, L) be the hypersurfaces of (M^n, L^*) and (M^n, L) given by the same equation (3.1). If b_l is tangential to the hypersurface(M^{n-1}, L), and (M^n_x, g_x), $(M^n_x\, g^*_x)$, (M^{n-1}, g^*_x), (M^{n-1}, g_x), are the tangent Riemannian space to (M^n, L), (M^n, L^*) (M^{n-1}, L) (M^{n-1}, L^*) respectively, then we have the following :

(i) The second fundamental forms of $(M^{n-1}_x\, g_x)$ and $(M^{n-1}_x, g^*_x$)are proportional.

(ii) Every asymptotic direction of (M^{n-1}_x, g_x) is also an asymptotic direction of $(M^{n-1}_x, g^*_x$).

(iii) The rth fundamental tensors of (M^{n-1}_x, g_x) and $(M^{n-1}_x, g^*_x$) are related by

$$C^*_{(r)\alpha\beta} = p^{(3-r)/2} [C_{(r)\alpha\beta} - \sum_{m=2}^{r-1} X_{(m)\beta}\, Q_{(r+1-m)\alpha}]\ (3 \leqslant r \leqslant n) \qquad ...(4.4)$$

where

$$X_{(m)\alpha} = P \sqrt{2q\lambda}\; C_{(m)\alpha\delta}\, b^\delta\ (2 \leqslant m \leqslant n-1) \qquad ...(4.5)$$

$$Y_{(m)} = P \sqrt{2q\lambda}\; X_{(m)\delta}\, b^\delta\ (2 \leqslant m \leqslant n-1) \qquad ...(4.5b)$$

$$Q_{(2)\alpha} = X_{(2)\alpha} \qquad ..(4.5c)$$

$$Q_{(r)\alpha} = X_{(r)\alpha} - \sum_{m=2}^{r-1} Y_{(m)}\, Q_{(r+1-m)\alpha}\ (3 \leqslant r \leqslant n-1). \qquad ...(4.5d)$$

PROOF : (i) Since $b_i N^i = 0$ implies $m_i N^i = 0$, from (2.4), (3.11) and (4.3) it follows that

$$M^*_{\beta\gamma} = p^{1/2} M_{\beta\gamma}. \qquad \ldots(4.6)$$

This proves (i).

(ii) A direction t^α for which $M_{\alpha\beta}\, t^\alpha t^\beta = 0$ is said to be an asymptotic direction. In view of this definition and (4.6) we get (ii).

(iii) The validity of relation (4.4) is established by induction. Since C_{ijk} is an indicatory tensor, from (3.11) it follows that $M_{\beta\gamma}\, l^\gamma = 0$. Hence from (3.12) we have

$$C_{(r)\beta\gamma}\, l^\gamma = 0 = C_{(r)\beta\gamma}\, l^\beta \; (2 \leqslant r \leqslant n). \qquad \ldots(4.7)$$

Hence from (4.5) (a), (c) and (d) we get

$$X_{(r)\alpha}\, l^\alpha = 0,\; Q_{(r)\alpha}\, l^\alpha = 0 \; (2 \leqslant r \leqslant n). \qquad \ldots(4.8)$$

Since $g^{*\alpha\beta} = g^{*ij}\; B^\alpha_i\; B^\beta_j$, from (2.2) and (4.6) we get

$$M^{*\delta}_{\beta} = p^{-1/2}\, [M^\delta_\beta + pq^2 \lambda\, M_{\alpha\beta}\, b^\alpha l^\delta - p\sqrt{2q\lambda}\; X_{(2)\beta}\, b^\delta] \qquad \ldots(4.9)$$

where $b^\alpha = B^\alpha_i\; b^i$. The relations (3.12), (4.5a) (4.6), (4.8) and (4.9) yield

$$C^*_{(3)\alpha\beta} = C_{(3)\alpha\beta} - X_{(2)\alpha}\, X_{(2)\beta}. \qquad \ldots(4.10)$$

From (4.5c) and (4.10) it is evident that (4.4) holds for $r = 3$. For a given fixed value of the integer s with $3 \leqslant s \leqslant n - 1$ we have

$$C^*_{(s+1)\alpha\beta} = C^*_{(s)\alpha\beta}\; M^{*\delta}_{\beta}. \qquad \ldots(4.11)$$

Now let us suppose that (4.4) is valid for $s = 3, 4, 5, \ldots r.$ so that we can write (4.11) in the form

$$C^*_{(s+1)\alpha\beta} = p^{(3-s)/2}\, [C_{(s)\alpha\delta} - \sum_{m=2}^{s-1} X_{(m)\delta}\, Q_{(s+1-m)\alpha}]\, M^{*\delta}_{\beta}$$

which in view of (4.5), (4.7), (4.8) and (4.9) gives

$$C^*_{(s+1)\alpha\beta} = p^{(2-s)/2}\, [C_{(s+1)\alpha\beta} - \sum_{m=2}^{s-1} X_{(m+1)}\, Q_{(s+1-m)\alpha}$$

(*equation continued on p.* 249)

$$- X_{(2)\beta} \{X_{(s)\alpha} - \sum_{m=2}^{s-1} Y_{(m)} Q_{(s+1-m)\alpha}\}]$$

$$= p^{(2-s)/2} [C_{(s+1)\alpha\beta} - \sum_{m=2}^{s} X_{(m)\beta} Q_{(s+2-m)\alpha}].$$

This shows that (4.4) is valid for $r = (s + 1)$, which completes the proof of (iii).

Remark 1 : If $q\lambda < 0$, the quantities $X_{(m)\alpha}$, $Y_{(m)}$ and $Q_{(m)\alpha}$ defined in (4.5) are complex quantities. However the quantities $C^{*}_{(r)\alpha\beta}$ are always real.

Remark 2 : Theorem (4.1) and Theorem 4.2 are also valid for general β-changes.

ACKNOWLEDGEMENT

The authors are thankful to the referee for helpful suggestions.

REFERENCES

1. M. Matsumoto, *Tensor N. S.* **22** (1971), 103-11.
2. M. Matsumoto, *J. Math. Kyoto Univ.* **25** (1985), 107-44.
3. M. Matsumoto and S. Numata, *Tensor, N. S.* **33** (1979), 153-62.
4. M. Matsumotu and C. Shibata, *J. Math. Kyoto Univ.* **19** (1979), 301-14.
5. B. N. Prasad, *Indian J. pure appl. Math.* **10** (1978), 1068-75.
6. B. N. Prasad, *J. Math. Phys. Sci.* **13** (1979), 251-56.
7. H. Rund, *Tensor, N. S.* **22** (1971), 163-73.
8. H. Rund, *Ann. Math. Pure applic.* **88** (1971), 99-122.
9. C. Shibata, *J. Math. Kyoto Univ.* **24** (1984), 163-88.

Indian J. pure appl. Math., 20 (3) : 250-259, March 1989

COPURE-INJECTIVE MODULES*

V. A. HIREMATH

Department of Mathematics, University of Mysore, Mysore 570006, Karnataka

(*Received* 10 *November* 1987; *after revision* 22 *March* 1988)

In this paper we introduce the notion of a copure-injective module and study its properties over special types of rings. Our main structure theorem asserts that over an arbitrary ring R, the copure-injective right R-modules are precisely the direct summands of direct products of cofinitely related right R-modules. Over a commutative classical ring R the copure-injective R-modules are precisely the pure-injective R-modules. If R is a commutative ring then every pure-injective R-module is copure-injective. Over a Dedekind domain R, every copure homomorphic image of a copure-injective R-module is copure-injective. Finally we derive the analogue of Schanuel's lemma for copure short exact sequences and copure-injective modules.

Throughout this paper by a ring R we mean an associative ring with identity and by an R-module we mean an unitary right R-module while mod-R stands for the category of all right R-modules and R-homomorphisms.

Definitions 1—(i) An R-module M is said to be 'finitely embedded' (Vamos[19], p. 643) (or 'cofinitely generated' (Jans[10], p. 588)) if $E(M) = E(S_1) \oplus E(S_2) \oplus \ldots \otimes E(S_n)$, where $S_1, S_2, \ldots, S_n$ are simple R-modules (here $E(X)$ denotes the injective hull of an R-module X).

(ii) An R-module M is said to be 'cofree' (Hiremath[7], Definition 6) if M is isomorphic to $\Pi\,\{E(S_\alpha) : S_\alpha$ is a simple R-module, $\alpha \in \Lambda\}$ for some index set Λ.

(iii) An R-module A is said to be 'cofinitely related' (Hiremath[7], Definition 14) if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B cofinitely generated, cofree and C cofinitely generated.

(iv) A short exact sequence of R-modules is said to be 'copure' (Hiremath[8], Definition 3) if every cofinitely related R-module is injective relative to this sequence.

(v) A ring R is said to be 'right co-noetherian' (Jans[10], p. 588) if every homomorphic image of a cofinitely generated R-module is cofinitely generated.

(vi) A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is said to be 'pure' (Warfield[21], p. 703) if for every left R-module M, the induced map $f \otimes l_M : A \otimes_R M \to B \otimes_R M$ of abelian groups is a monomorphism.

*This work is based on the author's doctoral dissertation and was supported by the University Grants Commission and Madurai Kamaraj University.

(vii) An R-module is said to be 'pure-injective' (Warfield[21], p. 703) if it is injective relative to each pure short exact sequence of R-modules.

Definition 2—An R-module is said to be copure-injective if it is injective relative to every copure short exact sequence of R-modules.

From the definition we have following easy consequences.

Proposition 3—(i) Every injective R-module is copure-injective.

(ii) Every cofinitely related R-module is copure-injective.

(iii) A copure short exact sequence $0 \to A \to B \to C \to 0$ of R-modules with A copure-injective, splits.

(iv) Every direct summand of a copure-injective R-module is copure-injective.

(v) Every direct product of copure-injective R-modules is copure-injective.

Remark 4 : A copure-injective R-module need not be injective.

Example—Let $G = Z(n)$ be a cyclic group of order n. Since G is finite it is cofinitely generated and hence, cofinitely related as a Z-module (Hiremath[7], p, 5 and Proposition 17). So, by Proposition 3 (ii), G is copure-injective as a Z-module. But we know that G is not injective as a Z-module.

Proposition 5—For a ring R the following conditions are equivalent:

(i) R is a right V-ring;

(ii) every cofinitely related R-module is injective;

(iii) every short exact sequence of R-modules is copure;

(iv) every copure-injective R-modules is injective.

Pooof : (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow by Proposition 5 of Hiremath[8]. (iii) $\Rightarrow$ (iv) is obvious and (iv) $\Rightarrow$ (ii) follows by Proposition 3 (ii).

Remark : A direct sum of (copure—) injective R-modules need not be copure-injective (see example (i) below). We know that over a right noetherian ring R every direct sum of injective R-modules is injective (Faith[3], Proposition 6.5). But over a right noetherian ring R (even when R is a Dedekind domain) a direct sum of copure-injective R-modules need not be copure-injective (see example (ii)).

Examples—(i) Let $R = \prod_{n=1}^{\infty} Z/p_n Z$, where $\{p_1, p_2, \ldots, p_n, \ldots\}$ is the set of all positive primes. Since R is a Von Neumann regular ring, R is a V-ring by Theorem 6 of Rosenberg and Zelinsky[10]. Let, for $n = 1,2, \ldots$, $Q_n = \prod_{i=1}^{n} Z/p_i Z$. Then $Q_1 \subsetneq$

$\mathscr{Q}_2 \underset{+}{\subset} \cdots \underset{+}{\subset} \mathscr{Q}_n \underset{+}{\subset} \cdots$ is a strictly ascending chain of ideals of R. Then, by Proposition 6.5 of Faith[3], $E = \overset{\infty}{\underset{n=1}{\oplus}} E(R/\mathscr{Q}_n)$ is not injective and so, not copure-injective by Proposition 5, as R is a V-ring.

(ii) Let $G = \overset{\infty}{\underset{n=1}{\oplus}} Z/p_n Z$ where $\{p_1, p_2, \ldots, p_n, \ldots\}$ is the set of all positive primes. Since each $Z/p_n Z$ is finite, it is cofinitely related as a Z-module by the remark on p. 5 and Proposition 17 of Hiremath[7] and so, by Proposition 3 (ii), each $Z/p_n Z$ is copure-injective as a Z-module. Clearly G is a pure submodule of the Z-module $H = \overset{\infty}{\underset{n=1}{\Pi}} Z/p_n Z$. Hence, by Proposition 12 of Hiremath[8], G is a copure submodule of the Z-module H as Z is a neotherian ring. If G were copure-injective, then by Proposition 3 (iii), G would be a direct summand of H which is absurd. So G is not copure-injective as a Z-module.

We recall (Faith[4], p. 254) that a category $\mathscr{C}$ is 'locally small' if the equivalence class of subobjects of any object of $\mathscr{C}$ is a set and $\mathscr{C}$ is 'colocally small' if the dual category $\mathscr{C}^*$ of $\mathscr{C}$ is locally small. Since the category mod-R is colocally small (Faith[4], Exercise 5.27.4), the isomorphisms closed class of cyclic R-modules, distinct up to isomorphisms, is a set. Hence the isomorphism closed class $\mathscr{S}$ of simple R-modules, distinct up to isomorphisms, is a set. Since mod-R is a locally small category (Faith[4], Exercise 5.27.4), the isomorphism closed class of subobjects of the R-module $A = \Pi \{E(S) : S \in \mathscr{S}\}$ (distinct upto isomorphisms) is a set. Since any cofinitely generated (and hence, cofinitely related) R-module is isomorphic to submodule of A, the isomorphim closed class C_r^* of cofinitely related R-modules, distinct up to isomorphisms, is a set.

We now prove that for any ring R, there are enough copure-injective R-modules in the sense that every R-module can be embedded in a copure-injective R-module as a copure submodule by applying Proposition 3 to :

Theorem 7—Every R-module can be embedded ais a copure submodule in a direct product of cofinitely related R-modules.

PROOF : Let C_r^* be the isomorphism closed class of all cofinitely related R-modules distinct up to isomorphisms. We have seen above that C_r^* is a set.

Let A be any R-module and let

$$Q = \Pi \{M^{\operatorname{Hom}_R(A,M)} : M \in C_r^*\}.$$

Define a map

$$\phi : A \to Q \text{ by}$$

$$\phi(a) = ((f(a))_{f \in \mathrm{Hom}_R(A, M)})_{M \in C_r^*}, \ a \in A.$$

Clearly ϕ is an R-homomorphism. We claim that ϕ is a copure monomorphism. Let $0 \neq a \in A$. Let $\mathcal{M}$ be a maximal right ideal of R containing $\mathrm{ann}_R(a)$. Then $S = R/\mathcal{M}$ is cofinitely generated and hence $E(S) \in C_r^*$. Define a map $\alpha : aR \to S$ by $\alpha(ar) = r + \mathcal{M}$, $r \in R$. Then α extends to a homomorphism $f : A \to E(S)$. Since $f(a) = 1 + \mathcal{M} \neq 0$, $\phi(a) \neq 0$, proving that ϕ is a monomorphism.

To prove that ϕ is copure, let $M \in C_r^*$ and $f \in \mathrm{Hom}_R(A, M)$. Then clearly $f = p\phi$ where $p : Q \to M$ is the composition $\pi_f \circ \pi_M$ of the usual projections π_M and π_f.

We now have the following structure theorem for copure-injective modules.

Theorem 8—An R-module M is copure-injective if and only if it is a direct summand of a direct product of cofinitely related R-modules.

PROOF: Necessity follows from Theorem 7 and Proposition 3 (iii) and sufficiency from (ii), (iv) and (v) of Proposition 3.

We now compare copure-injectivity with pure-injectivity. For this we first observe that from the adjoint isomorphism of Hom and $\otimes$ one can easily deduce that for a left R-module M, the canonical R-module $M^* = \mathrm{Hom}_Z(M, Q/Z)$ is pure-injective.

Proposition 9—Over a commutative ring R every pure-injective R-module is copure-injective.

PROOF: This follows from the fact that for a commutative ring R, copurity implies purity (Hiremath[9], Proposition 13).

Remark 10: A pure--injective R-module need not be copure-injective.

Example—Cozzens[2] has constructed a ring $R = k[x, D]$ of differentiable polynomials in a single indeterminate x over a universal field k with a derivation D (where multiplication is given by $ax = xa + D(a)$, $a \in k$). Cozzens has proved that R is a right V-ring but not a field. Since x is not invertible in R, Rx is not a pure left ideal of R and hence R/Rx is not flat as a left R-module. So by the Corollary on p. 131 of Lambek[12], $(R/Rx)^* = \mathrm{Hom}_Z(R/Rx, Q/Z)$ is not injective as an R-module whence, by Proposition 5, $(R/Rx)^*$ is not copure-injective. But $(R/Rx)^*$ is pure-injective.

Proposition 11—If R is a commutative (co-) noetherian ring then every copure-injective R-module is pure-injective.

PROOF : Since over a commutative (co-) noetherian ring R, every pure submodule of an R-module is copure (Hiremath[8], Proposition 12), the proposition follows.

We now prove a proposition which dualizes (i) $\Leftrightarrow$ (iv) of Proposition 5 :

Proposition 12—A ring R is a Von Neumann regular if and only if every pure-injective R-module is injective.

PROOF : Since for a Von Neumann regular ring R, every short exact sequence of R-modules is pure, by Theorem 11.24 of Faith[4] the 'only if part follows.

For the 'if' part, suppose that every pure-injective R-module is injective. To prove that R is Von Neumann regular, we need only prove, by Theorem 11.24 of Faith[4], that every left R-module M is flat. Indeed, since the R-module $M^* = \mathrm{Hom}_Z(M, Q/Z)$ is pure-injective, it is injective by hypothesis. It then follows. from the Corollory on on p. 131 of Lambek[12] that M is flat as a left R-module proving that R is Von Neumann regular.

Remark : A copure-injective R-module need not be pure-injective.

Example—Let V be a countably infinite dimensional vector space over the field Q of rational numbers. Let R be the ring of linear operators of V. Then R is a Von Neumann regular ring. Let $\mathcal{M}$ be a maximal right ideal of R containing the two-sided ideal I of R of all elements of R of finite rank, Then, by Theorem 1 of Osofsky[15], $S = R/\mathcal{M}$ is not injective as an R-module. Let $x \in E(S) \setminus S$ and let A be a submodule of $E(S)$ maximal with respect to $S \subseteq A$ and $x \notin A$. Then $E(S)/A$ is subdirectly irreducible and hence, is cofinitely generated. So A is cofinitely related and by Proposition 3 (ii), A is copure-injective. Since A is not a direct summand of $E(S)$, A is not injective and so not pure-injective by Proposition 12.

From Propositions 9 and 11 we have :

Corollary 14—If R is commutative (co-) noetherian ring then R-module is pure-injective if and only if it is copure-injective.

We now recall the following Definitions.

(i) (Warfied[21], p. 707-708). Let I be any index set and let M be any R-module. Let M^I be the Ith cartesian power of M. For any finite subset I^* of I, and elements $r_i \in R$ $(i \in I^*)$, we define a group homomorphism (which will be a module homomorphism if R is commutive)

$\phi : M^I \to M$ by $\phi(x) = \sum_{i \in I} r_i x_i$. By a linear equation over R in M we mean a pair (ϕ, m) where ϕ is a homomorphism of the type defined above and $m \in M$. The solution set of this equation $S(\phi, m)$ is the set of all elements $x \in M^I$ such that $\phi(x) = m$.

A family of linear equations over R in M is said to be finitely soluble if the corresp onding sets $S(\phi, m)$ have finite intersection property. An R-module M is said to

be algebraically compact if every finitely soluble system of linear equations over R in M has a simultaneous solution.

(ii) An R-module M is said to be linearly compact (in the discrete topology) (Warfield[21], p. 711 and Vamos[20], p. 115) if every family of cosets in M with finite intersection property has nonempty intersection.

(iii) A commutative ring R is said to be classical (Vamos[20], p. 121) if $E(S)$ is linearly compact for every simple R-module S (or equivalently, every cofinitely generated R-module is linearly compact).

We now generalize the Proposition 11 to commutative classical rings by noting that commutative (co-) noetherian rings are classical (Vamos[19], Theorem 2 and Vamos[20], Proposition 4.1).

Proposition 15— Over a commutative classical ring R, every copure injective R-module is pure-injective.

PROOF : This follows from Proposition 9 of Warfield[21] that over a commutative ring R, every linearly compact R-module is algebraically compact and hence, pure-injective (Warfield[21], Theorem 2), from Theorem 8 and the facts that pure-injective R-modules are closed under taking arbitrary direct products and direct summands.

Corollary 16—For a commutative classical ring R, every pure shart exact sequence of R-modules is copure.

Corollary 17—For a commutative classical ring R. an R-module is pure-injective if and only if it is copure-injective.

PROOF : This follows from Propositions 9 and 15.

We next characterize the rings R for which every R-module is copure-injective.

From Theorem 8 we have :

Corollary 18—For a ring R the following conditions are equivalent.

(i) Every R-module is copure-injective.

(ii) Every R-module is a direct summand of a direct product of cofinitely related R-modules.

(iii) Every copure short exact sequence of R-modules splits.

Fieldhouse[5] (p. 15) calls a ring R a 'right PDS ring' if every pure submodule of an R-module is a direct summand (or, equivalently (Fieldhouse[5], Theorem 10.1) every pure short exact sequence of R-modules splits). Similarly we call a ring R a right CDS ring if R satisfies the equivalent conditions of Corollary 18.

Since a ring R is a right PDS ring if and only if every R-module is pure-injective, we have from Proposition 9 and Corollary 17 :

Proposition 19—(i) If R is a commutative PDS ring then R is a CDS ring.

(ii) If R is a commutative classical ring then R is a PDS ring if and only if it is a CDS ring.

We now have, from Proposition 19 and Theorem 10.3 of Fieldhouse[5] :

Corollary 20—Every commutative classical CDS ring is artinian.

We do not know whether a commutative artinian ring is CDS. However, we have from Theorem 10.4 of Fieldhouse[5] :

Corollary 21—Every commutative uniserial ring, that is, a commutative artinian principal ideal ring, is a CDS ring.

Remark 22 : We know that Dedekind domain need not be a CDS (= PDS) ring (cf. Proposition 19 (ii)) (e. g. Z, the ring of integers, is not a CDS ring since Z is not algebraically compact as an abelian group (Fuchs[6], Chapter VII, § 38, Exercise 1)). But every proper homomorphic image of a Dedekind domain, being an artinian principal ideal ring (and hence a uniserial ring), is a CDS ring by Corollary 21.

Remark 23 : We know (Cartan and Eilenberg[1], Proposition 6.1) that a ring R is right hereditary if and only if every homonorphic image of an injective R-module is injective. But neither a homomorphic image of a copure-injective R-module over a right hereditary ring need be copure-injective (see example (i) below) nor a ring R for which every every homomorphic image of a copure-injective R-module is copure-injective need be right hereditary (see example (ii)).

Examples—(i) We know that Z is a hereditary ring and copure-injective Z-modules are precisely the algebraically compact Z-modules (Theorem 2 of Warfield[21] and Corollary 14). Since there exists an algebraically compact abelian group with a homomorphic image which is not algebraically compact, it follows that there exists a copure-injective Z-module with a homomorphic image which is not copure-injective.
(ii) $R = Z/(4)$ is an artinian principal ideal ring and hence is a uniserial ring. Then it is a CDS ring by Corollary 21. So, every R-module, in particular every homomorphic image of a copure-injective R-module, is copure-injective. But R is not a hereditary ring since the maximal ideal $(2)/(4)$ of the local ring R is not projective as it is not free as an R-module.

Remark 24 : We know (Fuchs[6], §38, Exercise 3) that every pure homomorphic image of an algebraically compact abelian group is algebraically compact. But, in general, a copure homomorphic image of a copure-injective R-module need not be copure-injective.

Example—Let $R = \prod_{p \in P} Z/pZ$ where P is the set of all positive primes. Since R is self-injective (Faith[4], Exercise 5.64.2), it is self-copure-injective. Since R is not semisimple artinian, there exists, by the Theorem of Osofsky[14], an ideal I of R such

that R/I is not injective as an R-module. Since R is a V-ring, R/I is not copure-injective as an R-module by Proposition 5.

We prove below, in Corollary 28, that every copure-homomorphic image of a copure-injective R-module over a Dedekind domain R is again copure-injective. Before proving this we take note of and derive a few results:

(i) Every finitely presented R-module is pure-projective.

(ii) Any R-module can be embedded in a pure short exact sequence $0 \to K \to P \to M \to 0$ of R-modules where P is pure-projective.

(iii) An R-module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented R-modules.

Next we recall (Kaplansky[11], p. 332) that if R is a commutative integral domain, then an R-module M is said to be decomposable if M is a direct sum of cyclic R-modules and finitely generated R-modules of rank one. Since over a Dedekind domain R, every finitely generated torsion-free R-module is projective (Cartan and Eilenberg[1], Chapter VII, Proposition 4.1 and a remark on p. 134), it follows that every decomposable R-module, over a Dedekind domain R, is a direct sum of cyclic R-modules and a projective R-module and hence it is pure-projective by (i) and (ii) of the results stated above.

Proposition 25—Over a Dedekind domain R every submodule of pure-projective R-module is pure-projective.

PROOF : We first observe that every pure-projective R-module is decomposable. This follows from : (a) Steinitz's theorem (Steinitz[17]) that every finitely generated R-module over a Dedekind domain R is decomposable; (b) the decomposability is a hereditary property for a Dedekind domain (Kaplansky[11], Theorem 4) and (c) form (iii) of the properties of pure-projective modules stated above.

From this observation and the observation made just before this proposition it follows that pure-projectivity and decomposability are equivalent for a Dedekind domain. The proposition now follows from Theorem 4 of Kaplansky[11].

Proposition 26—An R-module Q is pure-injective if and only if Q is injective relative to each pure short exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B pure-projective.

PROOF : This follows by using arguments similar to the Proposition 5.2 of Chapter *I* of Cartan and Eilenberg[1] and the fact that the class of all pure short exact sequences of R-modules form a 'Proper Class' in the sence of MacLane[13] (p. 367) (Stenström[18], Propositions 2.2 and 9.1).

By Propositions 25 and 26 and using arguments similar to Theorem 5.4 of Chapter I of Cartan and Eilenberg[1], we have :

Proposition 27—Every pure homomorphic image of a pure-injective R-module over a Dedekind domain R is pure-injective.

Since, by Theorem 20 of Hiremath[8], purity and copurity are equivalent for a Dedekind domain we have, from Corollary 14 and Proposition 27 :

Corollary 28—For a Dedekind domain R, every copure homomorphic image of a copure-injective R-module is copure-injective.

Finally we prove the analogue of Schanuel's lemma for copure short exact sequences and copure-injective modules. We give the proof for the sake of completeness.

Proposition 22—Let A be an R-module and let $0 \to A \xrightarrow{f} Q \xrightarrow{g} B \to 0$ and $0 \to A \xrightarrow{f'} Q' \xrightarrow{g'} B' \to 0$ be copure short exact sequences of R-modules with Q, Q' copure-injective. Then $Q \oplus B'$ and $Q' \oplus B$ are isomorphic.

Proof : Considering the usual pushout $C = (Q \oplus Q')/K$ where $K = \{(f(a), -f'(a)) : a \in A\}$ and the natural maps h, h' and by defining k, k' in the obvious way we obtain the following commutative diagram of exact rows and columns :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "f'"'] & Q \arrow[r, "g"] \arrow[d, "h"'] & B \arrow[r] \arrow[d, "1_B"] & 0 \\
0 \arrow[r] & Q' \arrow[r, "h'"] \arrow[d, "g'"'] & C \arrow[r, "k"] \arrow[d, "k'"] & B \arrow[r] & 0 \\
 & B' \arrow[r, "1_{B'}"] \arrow[d] & B' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

We prove that the second row and the second column are copure. Let M be any cofinitely related R-module and let $\alpha : Q' \to M$ be any homomorphism. Then by the copurity of the first row there is a $\beta : Q \to M$ such that $\beta f = \alpha f'$. Define $\phi : C \to M$ by $\phi((x, x') + K) = \beta(x) + \alpha(x')$ for $(x, x') + K$ in C. Clearly ϕ is a well-defined homomorphism and $\phi h' = \alpha$. So the second row is copure. Similarly the copurity of of the second column follows.

Now by the copure-injectivity of Q, Q' and the copurity of the second row and the second column, both the second row and the second column split by Proposition 3 (iii). Hence C is isomorphic to both $Q \oplus B'$ and $Q' \oplus B$.

Acknowledgement

The author thanks Professor K. M. Rangaswamy for suggesting the problem, for his encouragement and for many useful discussions. The author also thanks the referee for his comments in the revision of this paper.

REFERENCES

1. H. Cartan and S. Eilenberg, *Homological Algebra*. Princeton University Press, 1956.
2. J. H. Cozzens, *Bull. Am. Math. Soc.* 76 (1970), 75–79.
3. C. Faith, *Lectures on Injective Modules and Quotient Rings*, Lecture notes in Mathematics. Vol. 49, Springer-Verlag, New York-Heidelberg-Berlin, 1967.
4. C. Faith *Algebra I : Rings, Modules and Categories*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
5. D. J. Fieldhouse, *Math. Ann.* **184** (1970), 1–18.
6. L. Fuchs *Infinite Abelian Groups*, Vol. I. Academic Press, New York-London, 1970.
7. V. A. Hiremath, *Acta Math. Acad. Sci. Hungar.* **39** (1-3) (1983), 1–9.
8. V. A. Hiremath, *Acta Math. Hungar.* **44** (1-2) (1984), 3–12.
9. V. A. Hiremath, *Proc. Edinburh Math. Soc.* **29** (1986), 289–98.
10. J. P. Jans, *J. London Math. Soc.* **1** (2) (1969), 588–90.
11. I. Kaplansky, *Trans. Am. Math. Soc.* **72** (1952), 327–40.
12. J. Lambek, *Lectures on Rings and Modules*. Blaisdell Publishing Company.
13. S. MacLane *Homology*. Springer-Verlag, Berlin-Heidelberg-New York, 1967.
14. B. L. Osofsky, *Pacific J. Math* 14 (1964), 645–50.
15. B. L. Osofsky, *Proc. Amer Math. Soc.* **17** (1966), 247–53.
16. A. Rosenberg and D. Zelinsky, *Z. Math.* **70** (1959), 372–80.
17. E. Steinitz, *Math. Ann.* **71** (1912), 328–54
18. B. Stenström, *Arkiv Mat.* 7 (1867), 159–71.
19. P. Vamos, *J. London Math. Soc.* **43** (1968), 643–46.
20. P. Vamos, *J. Algebra* **34** (1975), 114–29.
21. R. B. Warfield (Jr.), *Pacific J. Math.* **28** (1969), 699–719.

Indian J. pure appl. Math., 20 (3): 260–266, March 1989

INCLUSION THEOREMS ON MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES OVER NON-ARCHIMEDIAN FIELDS IV

D. SOMASUNDARAM

Department of Mathematics, Madras University P. G. Centre
Salem 636011, *Tamil Nadu*

(*Received* 23 *March* 1988)

Necessary and sufficient conditions for an infinite matrix defined over a field K with non-trivial non-archimedian valuation to transform $\Gamma^*(K)$ into $C_0(K)$ and $\Gamma(K)$ to $\Gamma^*(K)$ are investigated. Tne sequence spaces $C_0(K)$, $\Gamma(K)$ and $\Gamma^*(K)$ are all defined over such a field K.

1. INTRODUCTION

The object of the present paper is to obtain some inclusion theorems of certain sequence spaces over non-archimedian fields which were not considered earlier[3–5]. In section 2 we shall describe the required preliminaries, whereas in section 4, we shall prove the main theorem of this paper.

2. ENTIRE FUNCTIONS OVER K

Let K be a non-archimedian non-trivially valued field which is complete under the metric of real valuation. Let N_k be defined as

$$N_k = \{ \mid x \mid : x \in K\}.$$

Let $\alpha(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ be a power series with coefficients in K. As noted by Raghunathan[2] (p. 517), $\alpha(x)$ is an entire function over K if and only if

$$\mid a_n \mid^{1/n} \to 0 \text{ as } n \to \infty.$$

If $\alpha(x)$ is an entire function, let

$$M(R) = \underset{\mid x \mid = R}{\text{Max}} \mid \alpha(x) \mid, \; (R \in N_k)$$

Then by a known result[1] (p. 85), we have

$$M(R) = \sup_n [\mid a_n \mid R^n, n \geqslant 0] \quad \ldots(2.1)$$

where

$$\alpha = \sum_{n=0}^{\infty} a_n x^n.$$

3. Definition of Different Sequfncf Spaces

In what follows, the notion of convergence and boundedness will be in relation to the metric of valuation of the field.

$C_0(K)$: The set of all null sequences $x = (x_n)$

$\Gamma(K)$: The set of all sequences $x = (x_n)$ with

$|x_n|^{1/n} \to 0$ as $n \to \infty$.

$\Gamma^*(K)$: The set of all sequences $x = (x_n)$ with

$\{|x_n|^{1/n}\}$ bounded.

If $x = (x_n)$ is a sequence over K, let us define $\|x\| = \sup\limits_{n \geqslant 1} |x_n|$. The norm is evidently non-archimedian in the sense that it satisfies stronger form of triangle inequality. With this as the norm $C(K)$ is a non-archimedian Banach space.

Let $x = (x_n) \in K$. Let $|x| = \text{Sup}\{|x_n|^{1/n}, n \geqslant 1\}$. Then $|x|$ satisfies the following conditions.

(i) $|x| > 0$, $|x| = 0$ if and only if $x = (0, 0, \ldots\ldots)$ where 0 is the zero element of the field.

(ii) $|x + y| < \text{Max}\{|x|, |y|\}$

(iii) $|tx| < A(t)|x|$, $t \in K$, $A(t) = \text{Max}\{1, |t|\}$.

Hence $|x - y|$ defines a metric in the set of all sequences $\Gamma(K)$. So $\Gamma(K)$ is a metric space defined over K, the topology being defined by the metric given above. Raghunathan[2] (p. 518) has proved that $\Gamma(K)$ is a complete linear metric space which is totally disconnected.

According to Raghunathan[2] every continuous linear functional $f(x)$ defined for $x \in \Gamma(K)$ is of the form $f(x) = \Sigma C_n x_n$, $x = (x_n)$ where $\{|C_n|^{1/n}\}$ is bounded. Hence $\Gamma^*(K)$ is identified as the dual space of $\Gamma(K)$. It was known from Raghunathan[2] (p. 524), $\Gamma^*(K)$ is a complete metric space which is not linear.

4. Matrix Transformation of $\Gamma^*(K)$ into $C_0(K)$

Let us consider the matrix transformation

$$y_n = \sum_{p=1}^{\infty} a_{np}\, x_p,\ n = 1, 2, 3, \ldots, \text{ and } a_{np} \in K. \qquad \ldots(4.1)$$

Theorem 1— A necessary and sufficient condition that $(y_n) \in C_0(K)$ whenever $(x_n) \in \Gamma^*(K)$ is that

$\{f_n(x)\}$ is bounded uniformly on every finite circle

$$| x | \leqslant R, R \in N_k \qquad \ldots(4.2)$$

where

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x_p, n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

is a sequence of entire functions over K.

Proof : The condition is sufficient.

Let

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x_p, n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

be uniformly bounded on every finite circle $| x | \leqslant R$, $R \in N_k$. Then the condition (4.2) implies that there exists a constant $M(R)$ such that $| f_n(x) | \leqslant M(R)$ for every x with the property that $| x | \leqslant R$, $R \in N_k$. By using (2.1) and the condition (4.2), we have

$$| a_{np} | R^p \leqslant M(R) \text{ for each fixed } p. \qquad \ldots(4.3)$$

Since $(x_p) \in \Gamma^*(K)$, there is a constant c such that

$$| x_p |^{1/2} \leqslant c \text{ so that } | x_p | \leqslant c^p \text{ for all } p. \qquad \ldots(4.4)$$

From (4.1),

$$| y_n | \leqslant \sup_{1 \leqslant p \leqslant \infty} | a_{np} | \; | x_p | .$$

By using (4.3) and (4.4) in the above, we have

$$| y_n | \leqslant \frac{M(R)}{R^p} c^p.$$

Choose $R = \mu c$ where $\mu \in N_k$ and $\mu > 1$.

Then $| y_n | \leqslant \dfrac{M(\mu c)}{\mu^p c^p} c^p = \dfrac{M(\mu c)}{\mu^p} \leqslant \epsilon$ for large p so that $y_n \to 0$ as $n \to \infty$.

The condition is necessary.

Since $f_n(x) = \Sigma a_{np} x^p$ is defined for every n and $x = R$, the series is convergent for $| x | = R$.

$$| a_{np} z^p | = | a_{np} | R^p \to 0 \text{ as } p \to \infty. \qquad \ldots(4.5)$$

When we consider the special sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ where 1 is in the pth place,

$$y_n = a_{np}, n = 1, 2, 3, \ldots .$$

Since $(y_n) \in C_0(K)$, we have for each fixed p

$$a_{np} \to 0 \text{ as } n \to \infty. \qquad \text{.. (4.6)}$$

If the condition (4.2) is not satisfied, there is some finite circle $|x| \leqslant R$, $R \in N_k$ such that $\{f_n(x)\}$ is not bounded uniformly on $|x| = R$. Since $|x| = R$ is bounded in the metric of valuation, we can find a sequence (x_n) such that for $|x_n| = R$, $f_n(x_n) \to \infty$ as $n \to \infty$, through a subsequence of values of n.

Using this assumption, we shall construct a sequence $(x_p) \in \Gamma^*(R)$ with the additional condition $|x_p| = R^p$ for every p such that the corresponding (y_n) is not bounded. This proves the necessity of the condition.

Choosing n_1 such that

$$|f_{n_1}(x_{n_1})| > \epsilon \text{ for some } \epsilon > 0. \qquad \text{.. (4.7)}$$

Since $|x_{n_1}| = R$, (2.1) and (4.7) together imply,

$$\sup_{1 \leqslant p < \infty} |a_{n_1 p}| R^p > \epsilon. \qquad \text{.. (4.8)}$$

By using (4.5), we have

$$\sup_{p_{n_1}+1 \leqslant p < \infty} |a_{n_1 p}| R^p < \frac{\epsilon}{2} \text{ for } p \geqslant p_{n_1}. \qquad \text{.. (4.9)}$$

From (4.8) and (4.9), we get

$$\sup_{1 \leqslant p \leqslant p_{n_1}} |a_{n_1 p}| R^p > \epsilon.$$

Hence there is a p_1 in $1 \leqslant p \leqslant p_{n_1}$ such that

$$|a_{n_1 p_1}| R^p > \epsilon. \qquad \text{...(4.10)}$$

Now choose x_p in $1 \leqslant p \leqslant p_{n_1}$ as follows.

$$x_p = \begin{cases} x^p \text{ for } p = p_1 \text{ and } |x| = R \\ 0 \text{ for all } p \text{ in } 1 \leqslant p \leqslant p_{n_1}. \end{cases} \qquad \text{...(4.11)}$$

Then

$$y_{n_1} = \sum_{p=1}^{p_{n_1}} a_{np}\, x_p + \sum_{p_{n_1}+1}^{\infty} a_{n_1}\, x_p. \qquad \text{...(4.12)}$$

$$\left| \sum_{p_{n_1}+1}^{\infty} a_{n_1 p}\, x_p \right| \leqslant \sup_{p_{n_1}+1 \leqslant p < \infty} |a_{n_1 p}| \, |x_p| < \frac{\epsilon}{2} \text{ by (4.9).} \qquad \text{...(4.13)}$$

Hence we shall have from (4.12)

$$|\sum_{p=1}^{p=p_{n_1}} a_{n_1 p}\, x_p| = |a_{n_1 p}\, x_{p_1}| \leqslant \text{Max}\ \{ |y_{n_1}|, |\sum_{p_{n_1}+1}^{\infty} a_{n_1 p}\, x_p| \}.$$

By (4.10), (4.11) and (4.13), we have

$$\epsilon < \text{Max}\ \{ |y_{n_1}|, \frac{\epsilon}{2} \}$$

Therefore from the above, we have

$$|y_{n_1}| > \epsilon.$$

Since $a_{n_p} \to 0$ as $n \to \infty$ for each fixed p. So $|a_{np}|\, R^p \to 0$ as $n \to \infty$ for each fixed p

$$|a_{np}|\, R^p < \epsilon \text{ for all } n > n_p.$$

Now choose $n_2 > n_1$ such that

$$\left.\begin{array}{l} \sup\limits_{1 \leqslant p < \infty} |a_{n_2 p}|\, R^p > \epsilon \\ \text{and} \\ \sup\limits_{1 \leqslant p \leqslant p_{n_1}} |a_{n_2 p}|\, R^p < \dfrac{\epsilon}{2}. \end{array}\right\} \qquad \ldots(4.14)$$

This is possible if n_2 is large enough such that

$$n_2 > \text{Max}\ (n_p) \text{ where } 1 \leqslant p \leqslant p_{n_1}.$$

Then there exists by (4.5) p_{n_2} greater than p_{n_1} such that

$$\sup_{p_{n_1}+1 \leqslant p < \infty} |a_{n_2 p}|\, R^p < \frac{\epsilon}{2}. \qquad \ldots(4.15)$$

Therefore from (4.14) and (4.15), we have

$$\sup_{1 \leqslant p \leqslant p_{n_2}} |a_{n_2 p}|\, R^p > \epsilon. \qquad \ldots(4.16)$$

Hence there exists a p_2 in $1 \leqslant p \leqslant p_{m_2}$ such that

$$|a_{n_2 p_2}|\, R^{p_2} > \epsilon. \qquad \ldots(4.17)$$

So p_2 chosen in (4.17) exceeds p_{n_1} by (4.14). Now choose x_p as follows.

$$x_p = \begin{cases} x^p \text{ when } p = p_2 \text{ with } |x| = R \\ 0 \text{ for all } p \text{ in } p_{n_1}+1 \leqslant p \leqslant p_{n_2} \end{cases} \qquad \ldots(4.18)$$

$$y_{n_2} = \sum_{p=1}^{p=p_{n_1}} a_{n_2 p}\, x_p + \sum_{p_{n_1}+1}^{p_{n_2}} a_{n_2 p}\, x_p + \sum_{p_{n_2}+1}^{\infty} a_{n_2 p}\, x^p$$

$$\sum_{p_{n_1}+1}^{p_{n_2}} a_{n_2 p}\, x_p = y_{n_2} - \sum_{p=1}^{p_{n_1}} a_{n_2 p}\, x_p - \sum_{p_{n_2}+1}^{\infty} a_{n_2 p}\, x_p.$$

By using (4.18) in the above, we get

$$\Big| \sum_{p_{n_1}+1}^{p_{n2}} a_{n_2 p}\, x_p \Big| = | a_{n_2 p_2} x_{p_2} |$$

$$\leqslant \text{Max} \Big\{ | y_{n2} | , \Big| \sum_{p=1}^{p_{n1}} a_{n_2 p}\, x_p \Big| , \Big| \sum_{p_{n_2}+1}^{\infty} a_{n_2 p}\, x_p \Big| \Big\} \qquad \ldots(4.19)$$

By (4.15), we have

$$\Big| \sum_{p_{n_2}+1}^{\infty} a_{n_2 p}\, x_p \Big| \leqslant \sup_{p_{n_2}+1 \leqslant p < \infty} | a_{n_2 p} | R^p < \frac{\epsilon}{2}. \qquad \ldots(4.20)$$

By (4.14), we have

$$\Big| \sum_{p=1}^{p_{n_1}} a_{n_2 p}\, x_p \Big| \leqslant \sup_{1 \leqslant p \leqslant p_{n_1}} | a_{n_2 p} | R^p < \frac{\epsilon}{2}.$$

Using (4.17), (4.18), (4.20), (4.21) in (4.19), we get

$$\epsilon < \text{Max} \left\{ | y_{n_2} | , \frac{\epsilon}{2}, \frac{\epsilon}{2} \right\}$$

This implies $| y_{n_2} | > \epsilon$.

Proceeding in the same manner, we can find n_k such that $| y_{n_k} | > \epsilon$. That is $| y_{n_k} |$ does not tend to zero as $n \to \infty$ through a subsequence of values of n. This shows that (y_n) does not belong to $C_0 (K)$, while the sequence $(x_p) \in \Gamma^* (K)$. This contradiction proves the necessity of the condition.

By using a method similar to the proof of the above Theorem 1, we can prove following theorem.

Theorem 2— A necessary and sufficient condition that $\{y_n\}$ should belong to $\Gamma (K)$ whenever (x_p) belong to $\Gamma^* (K)$ is that $| f_n (x) |^{1/n} \to 0$ as $n \to \infty$ uniformly on

every finite circle $| x | < R$, $R \in N_k$ where

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x^p, x\ n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

is a sequence of entire functions over K.

References

1. D. N. Lenskoi, Functions in non-archimedian valed fileds. *Russian, Saratov*, 1962.
2. T. T. Raghunathan, *Boll. U.M.I.* **4** (1968), 517–26.
3. D. Somasundaram, *Boll. U M I.* **6** (1970), 953–60
4. D. Somasundaram, *Indian J. pure appl. Math.* **5** (1974), 337–43.
5. D. Somasundaram, *Acta Ciencia Indica*, **4** (1978), 90–93.

Indian J. pure appl. Math., 20 (3) : 267-275, March 1989

COMMENCEMENT OF COUETTE FLOW IN OLDROYD LIQUID WITH HEAT SOURCES

G. C. DASH

Department of Mathematics, College of Basic Science and Humanities
O. U. A. T., Bhubaneswar 751003

AND

S. BISWAL

Department of Physics, Nayagarh College, Nayagrh 752069, *Puri*

(*Received* 11 *December* 1987; *after revision* 14 *June* 1988)

Commencement of Couette flow in Oldroyd liquid with heat sources has been analysed. Approximate solutions are derived by Galerkin technique. The effects of various parameters on the velocity field, the temperature field, the skin friction and the rate of heat transfer are discussed. It is observed that the flow is sensitive to the interactions of heat source and elasticity of the fluid.

1. INTRODUCTION

The problem of Couette flow between parallel walls has a rich stock of literature. Pai[1] has discussed the problem of unsteady Couette flow of a viscous inrompressible liquid between two plate walls occurring due to the sudden motion of one of the walls. The technique used for the solution of the equations is the principle of superposition and the solution occurs in the form of an infinite series of functions of the variables Y and t. Nanda[2] has studied the same problem assuming the walls to be porous. Then, it has been extended by Katagiri[3] and Muhuri[4] Rath *et al.*[5] who have solved the heat transfer problem in case of unsteady Couette flow between two parallel walls having different temperature.

Unsteady Couette flow in non-Newtonian fluids has enormous applications in technology. Therefore, many researchers have obtained the approximate solutions for the unsteady Couette flow caused in visco-elastic fluids with different wall velocities under different physical situations. Generalized plane Couette flow of one Oldroyd fluid with either suction or injection at the stationary wall has been discussed by Mishra[6]. Soundalgekar[7] has analysed the problem of plane Couette flow of Walters liquid B' with equal rate of injection at one wall and suction at the other (moving wall). Padhy[8] has analysed the commencement of unsteady Couette flow in second

order liquid. The object of the present paper is to study the commencement of unsteady Couette flow in case of Oldroyd liquid contained between two porous walls in the presence of heat sources.

2. Basic Equations

The visco-elastic fluid model considered here is the simplified form of Oldroyd's B-liquid[9] whose stress-strain relation is given by

$$\left[1 + \lambda_1 \frac{\delta}{\delta t}\right] P'^{ik} = 2\eta_0 \left[1 + \lambda_2 \frac{\delta}{\delta t}\right] e^{ik}. \qquad \ldots(1)$$

In this equation λ_1 and λ_2 are the time of relaxation and retardation respectively, η_0 is the co-efficient of viscosity, P'^{ik} and e^{ik} are respectively the stress and the strain rate tensors, $\delta/\delta t$ denotes the convective derivative and, for a contravariant second order tensor, b^{ik} is given by

$$\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m\, b^{ik}_{,m} + v^m_{,i}\, b^{mk} + v^m_{,k}\, b^{im}.$$

v^i being the velocity vector and

$$2e_{ik} = v_{i,k} + v_{k,i}.$$

Neglecting the second order terms in λ_1 and λ_2 in eqn. (1) and following Gieskus[10], we get

$$P'^{ik} = 2\eta_0\, e^{ik} - 2k_0 \frac{\delta}{\delta t}\, e^{ik} \qquad \ldots(2)$$

where

$$k_0 = \eta_0 (\lambda_1 - \lambda_2).$$

The momentum equations are given by

$$\rho \left[\frac{\partial v^i}{\partial t} + v^j\, v^i_{,j}\right] = -\,p_{,i} + p^{ij}_{,j} \qquad \ldots(3)$$

where ρ is the density of the medium and p an arbitrary isotropic pressure. The equation of continuity is

$$U^i_{,i} = 0 \qquad \ldots(4)$$

and the energy equation is given by

$$\rho\, C_p \frac{DT}{Dt} = k^* \nabla^2 T + \Phi \qquad \ldots(5)$$

where C_p is the specific heat at constant pressure, k^* the thermal conductivity and Φ the dissipation function prescribed by

$$\Phi = e_{ij}\, P_{ij}. \qquad \ldots(6)$$

3. Formulation of the Problem

Here, the unsteady Couette flow of an incompressible viscoelastic fluid begins with the sudden motion of the lower wall with time varying velocity At'^n where n is positive. The x'-axis is chosen along the lower wall and the y'-axis normal to it. The upper plane is specified by the equation $y' = L$ where the number L will be defined later. It is also supposed that the walls extend to infinity in both sides of the x'-axis. We consider the suction and injection velocity V at the walls to be constant. Then the velocity components u' and v' at any point (x', y') in the flow field compatible with the equation of continuity are given by

$$\left.\begin{aligned} u' &= u'(y, t) \\ v' &= V. \end{aligned}\right\} \qquad ...(7)$$

Under these conditions and also due to the shearing action of the fluid layers, the flow field and the temperature field with viscosity dissipation for an Oldroyd fluid are characterized by the following equations :

$$\rho\left(\frac{\partial u'}{\partial t'} + V\frac{\partial u'}{\partial y'}\right) = \eta_0\frac{\partial^2 u'}{\partial y'^2} - k_0\left(\frac{\partial^3 u'}{\partial y'^2 \partial t'} + V\frac{\partial^3 u'}{\partial y'^3}\right) \qquad ...(8)$$

$$\rho C_p\left(\frac{\partial \theta'}{\partial t'} + V\frac{\partial \theta'}{\partial y'}\right) = K^*\frac{\partial^2 \theta'}{\partial y'^2} + \eta_0\left(\frac{\partial u'}{\partial y'}\right)^2 - k_0\left[\frac{\partial^2 u'}{\partial y'\,\partial t'}\cdot\frac{\partial u'}{\partial y'} + V\frac{\partial u'}{\partial y'}\cdot\frac{\partial^2 u'}{\partial y'^2}\right] + S'(\theta' - \theta_L). \qquad ...(9)$$

The notations ρ, C_p, η_0, K_0, θ' and K^* used in the above equations are respectively the density, specific heat, co-efficient of viscosity, coefficient of elasticity, temperature at any point and thermal conductivity of the fluid. The last term in eqn. (9) represents the source-sink term with S' as its strength.

The adequate boundary conditions are

$$\left.\begin{aligned} &t' = 0 : u' = 0 \text{ for all } y' \\ &t' > 0 : \left.\begin{aligned} u' &= At'^n \text{ for } y' = 0 \\ u' &= 0 \quad \text{for } y' = L \end{aligned}\right\} \end{aligned}\right\} \qquad ...(10)$$

and

$$\left.\begin{aligned} &t' = 0 : \theta' = 0 \quad \text{for all } y' \\ &t' > 0 : \left.\begin{aligned} \frac{\partial \theta'}{\partial y'} &= 0 \text{ for } y' = 0 \\ \theta' &= \theta_L \text{ for } y' = L. \end{aligned}\right\} \end{aligned}\right\} \qquad ...(11)$$

The fact $\frac{\partial \theta'}{\partial y'} = 0$ when $y' = 0$ implies that the lower wall is a non conducting one. Now, we introduce the following dimensionless variables and parameters :

$$\left.\begin{aligned} & y = \frac{y'}{\sqrt{\nu_1 T}},\ t = \frac{t'}{T},\ u = \frac{u'}{AT^n} \\ & R = \frac{V\sqrt{T}}{\sqrt{\nu_1}},\ R_c = \frac{\lambda_1 - \lambda_2}{T},\ \sigma = \frac{\nu_1 \rho C_p}{K^*} \\ & E = \frac{A^2 T^{2n}}{C_p \theta_L},\ \theta = \frac{\theta' - \theta_L}{},\ S = \frac{4S' \nu_1}{V^2} \\ & \nu_1 = \frac{\eta_0}{\rho} \text{ and } L = \sqrt{\nu_1 T} \end{aligned}\right\} \qquad \ldots(12)$$

where T is some reference time, θ_L the temperature of the upper plate, and R, R_c E, σ and S are respectively the suction parameter, elastic parameter, Eckert number, Prandtl number non-dimensional source-parameter.

By the use of (12), eqns. (8) and (9) are reduced to non-dimensional form as follows :

$$\frac{\partial u}{\partial t} + R\frac{\partial^2 u}{\partial y} - \frac{\partial^2 u}{\partial y^2} + RR_c\frac{\partial^3 u}{\partial y^3} + R_c\frac{\partial^3 u}{\partial y^2\,\partial t} = 0 \qquad \ldots(13)$$

$$\frac{\partial \theta}{\partial t} + R\frac{\partial \theta}{\partial y} - \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2} + R_c\,E\,\sigma\left[R\,\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial y\,\partial t}\right]$$

$$+ \sigma E\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{4}R^2 S\theta = 0. \qquad \ldots(14)$$

The modified boundary conditions are

$$\left.\begin{aligned} & t = 0 : u = 0 \text{ for all } y \\ & t > 0 : \left.\begin{aligned} & u = t^n \text{ for } y = 0 \\ & u = 0 \text{ for } y = 1 \end{aligned}\right\} \end{aligned}\right\} \qquad \ldots(15)$$

and

$$\left.\begin{aligned} & t = 0 : \theta = 0 \text{ for all } y \\ & t > 0 : \begin{aligned} & \frac{\partial \theta}{\partial y} = 0 \text{ for } y = 0 \\ & \theta = 0 \text{ for } y = 1. \end{aligned} \end{aligned}\right\} \qquad \ldots(16)$$

3. Method of Solution

It is difficult to solve equations (13) and (14) which are of order 3. However, one of the possibility is to derive one solution for small values of R_c (< 1) and then we assume

$$u = u_0 + R_c\,u_1 + O\,(R_c^2). \qquad \ldots(17)$$

Substituting (17) in (13) and equating the like powers of R_c we get

$$\frac{\partial u_0}{\partial t} + R\frac{\partial u_0}{\partial y} - \frac{\partial^2 u_0}{\partial y^2} = 0 \qquad \text{...(18)}$$

$$\frac{\partial u_1}{\partial t} + R\left(\frac{\partial u_1}{\partial y^3} + \frac{\partial^3 u_0}{\partial y^3}\right) - \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^3 u_0}{\partial y^2\,\partial t} = 0. \qquad \text{...(19)}$$

The boundary conditions for eqns. (18) and (19) are obtained from eqns. (10) and (17) as follows :

$$\left.\begin{array}{l} t = 0 : u_0 = 0,\ u_1 = 0\ \forall\ y \\ t > 0 : \left.\begin{array}{l} u_0 = t^n,\ u_1 = 0 \text{ for } y = 0 \\ u_0 = 0\ u_1 = 0 \text{ for } y = 1 \end{array}\right\} \end{array}\right\}. \qquad \text{...(20)}$$

Equations (18), (19) and (14) subject to the conditions (20) and (11) are solved using Galarkin technique. For this the following infinite forms for u_0, u_1 and θ are proposed which satisfy the initial and boundary conditions.

$$u_0 \approx t^n(1-y) + a_1\, ty(1-y) + a_2\, t^2y^2(1-y^2) + \ldots$$

$$a_3\, t^3 y^3 (1-y)^3 + \ldots \qquad \text{...(21)}$$

$$u_1 \approx b_1\, ty(1-y) + b_2\, t^2 y^2 (1-y)^2 + b_3\, t^3y^3(1-y)^3 + \ldots \qquad \text{...(22)}$$

$$\theta \approx c_1\, t(1-y^2) + c_2\, t^2\, y(1-y^2)^2 + c_3\, t^3y^2(1-y^2)^3 + \ldots \qquad \text{...(23)}$$

where a_i, b_i and c_i ($i = 1, 2, 3$) are arbitrary constants to be determined. Substituting (21) – (23) into eqns. (18), (19) and (14) and neglectings a_i, b_i and c_i for $i \geqslant 4$, the defect functions D_{u_0}, D_{u_1} and D_θ are obtained which are minimized by the technique of orthogonalisation and result in the following nine double integrals

$$\left.\begin{array}{l} \int_0^1\int_0^1 D_{u_0}\, t^j y^j (1-y)^j\, dt\, dy = 0 \\ \int_0^1\int_0^1 D_{u_1}\, t^j y^j (1-y)^j\, dt\, dy = 0, \\ \int_0^1\int_0^1 D_\theta\, t^j j^{j-1} (1-y^2)^j\, dt\, dy = 0. \end{array}\right\} j = 1, 2, 3 \qquad \text{...(24)}$$

It is note worthy here that $t \in [0, 1]$.

Performing the integrations, we arrive at the nine algebraic equations involving the parameters a_j, b_j and c_j ($j = 1, 2, 3$). These nine linear equations are solved which give the constants a_j, b_j and c_j and hence the velocity field $u = u_0 + R_c\, u_1$ and the temperature field θ.

The non-dimensional shear stress τ_{xy} is given by

$$\tau_{xy} = \frac{P_{xy}\sqrt{T}}{\rho\sqrt{\nu_1}}\, AT^n = \frac{\partial u}{\partial y} - R_c\left\{R\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y\,\partial t}\right\}. \qquad \text{...(25)}$$

The shear stresses at the plates $\tau_0 = \tau_{xy} \mid_{y=0}$ and $\tau_1 = \tau_{xy} \mid_{y=1}$ are calculated. The rate of heat transfer at the plates

$$Nu_0 = - \left. \frac{\partial \theta}{\partial y} \right|_{y=0} \text{ and } Nu_1 = - \left. \frac{\partial \theta}{\partial y} \right|_{y=1} \text{ are also calculated.}$$

4. Results and Discussions

The flow characteristics are studied for two positive values of n, i. e.,

(i) $n = 1$, constant acceleration

(ii) $n = \frac{1}{2}$, variable acceleration.

It is observed from the Fig. 1 that the flow field exhibts almost a diametrically opposite behaviour for values of R_c greater than zero. It is further observed that the velocity

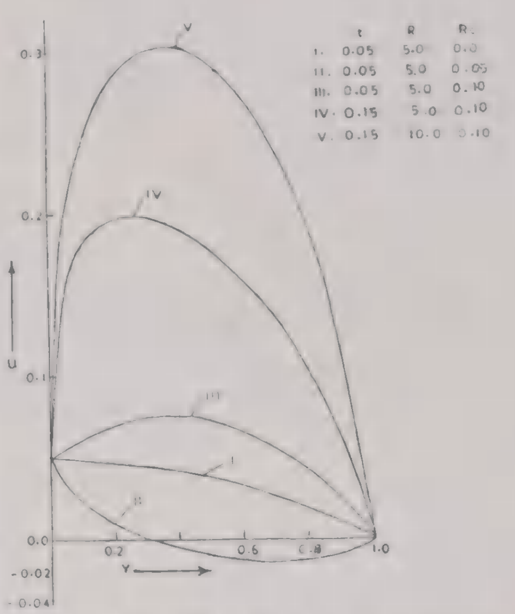

FIG. 1. Effect of R and R_c on velocity field, $n = 1.0$

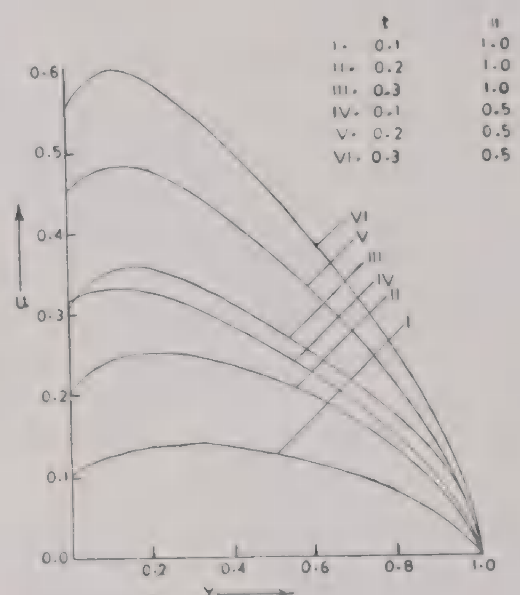

FIG. 2. Effect of t and n on velocity field, $R = 5.0$, $R_c = 0.1$

rises sharply with increasing values of t and this rise is quantitatively presented as 152%. To record the relative influence of the two parameters t and n on the flow pattern, it can be said from Fig. 2 that time plays a more effective role in increasing the velocity than n.

It is noticed from Fig. 3 that an increase in R_c is to decrease the temperature at all points and also an increase in time decreases it further.

Figure 4 shows that an increase in R is to increase the temperature at all points.

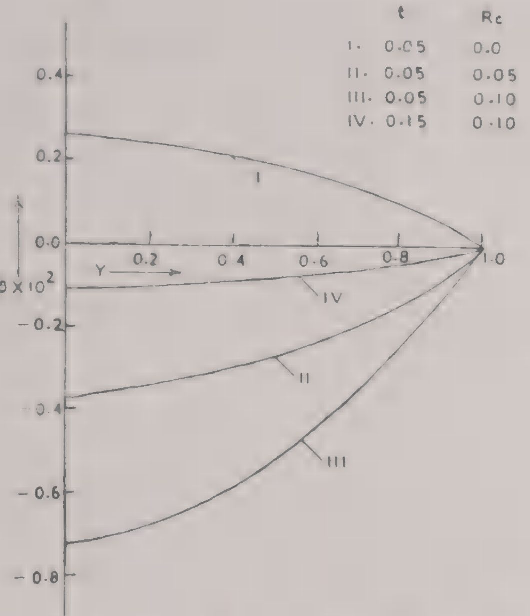

FIG. 3. Effect of R_c on temperature field, $R = 5.0$, $n = 1.0$, $S = 0.1$, $\sigma = 0.0$, $E = 0.01$

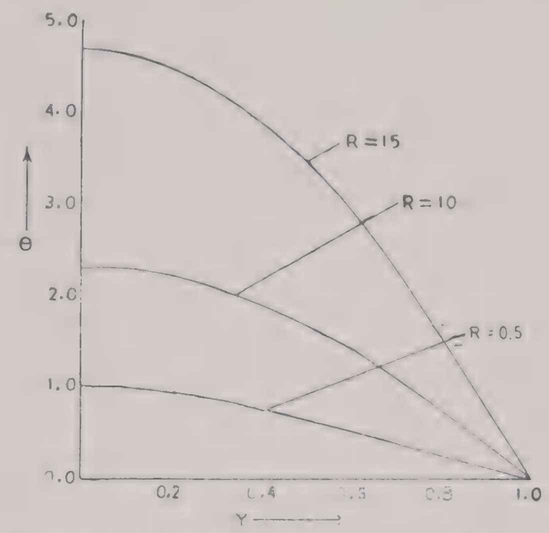

FIG. 4. Effect of R on temperature field, $R_c = 0.05$, $n = 1.0$, $S = 0.1$ $E = 0.001$, $\sigma = 0.5$

Figure 5 shows that the temperature field assumes negative values for $S = 1.0$ and $S = 0.5$. Further, it is observed that the temperature decreases with the decrease of source strength fiom 1.0 to 0.5 but for $S = 0.1$ it increases sharply.

Figure 6 shows that the shearing stress at the lower wall decreases with an increase in time for both variable and constant acceleration. It is also noticed that τ_0, in case of constant acceleration is greater than the variable acceleration for fixed

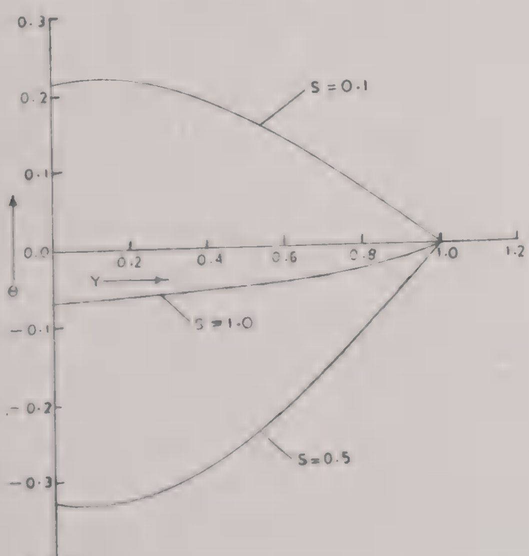

FIG. 5. Effect of S on temperature field, $R = 10.0$, $t = 0.05$, $R_c = 0.1$, $n = 1.0$, $\sigma = 0.2$, $E = 0.01$

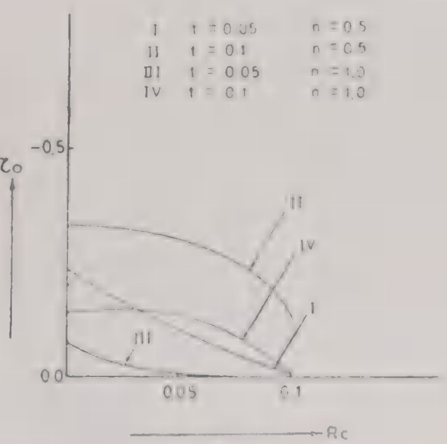

FIG. 6. Shearing stress at the lower wall, $R = 0.1$

value of t. Whereas the shearing stress at the upper wall τ_1 depicts an opposite behaviour for both t and n (Fig. 7). Further it is observed that the elasticity of the liquid increases both τ_0 and τ_1.

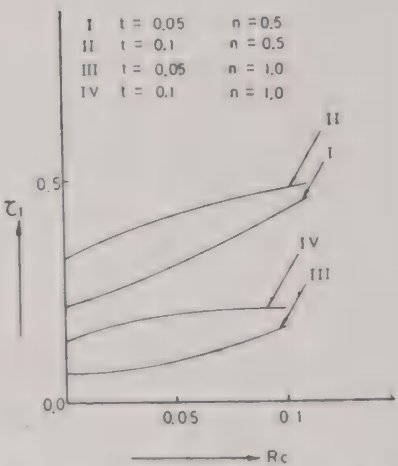

FIG. 7. Shearing stress at the upper wall, $R = 0.5$.

TABLE I

Values of the rates of heat transfer, Nu_0 and Nu_1 $S = 0.1$, $\sigma = 0.1$, $E = 0.01$

			n	0.5		1.0
R	t	R_c/n	Nu_0	Nu_1	Nu_0	Nu_1
	0.05	0.0	0.00004	0.00160	0.00003	0.00131
		0.05	0.00006	0.00194	0.00006	0.00193
		0.10	0.00004	0.00159	0.00003	0.00129
0.5						
		0.0	0.00019	0.00321	0.00375	0.00262
	0.10	0.05	0.00024	0.00389	0.00024	0.00387
		0.10	0.00019	0.00319	0.00015	0.00258
		0.0	0.00015	0.00628	0.00012	0.00520
5.0	0.05	0.05	−0.00112	−0.01647	−0.00095	−0.01449
		0.10	−0.00047	−0.01097	−0.00032	−0.00737

TABLE II

Showing the effect of S on the rates of heat transfer, Nu_0 and Nu_1 $t = 0.05$, $R_c = 0.1$, $n = 1.0$, $\sigma = 0.2$ $E = 0.01$

R	S	Nu_0	Nu_1
	0.1	0.00004	0.00140
0.5	0.5	0.00004	0.00140
	1.0	0.00004	0.00141
	0.1	0.00014	0.43116
10.5	0.5	0.00735	−0.65259
	1.0	0.00496	−0.13865

Table I describes the effects of R, R_c, t and n on the Nusselt number. Nu_0 and Nu_1 both increase when R_c takes the values from 0.0 to 0.05. The rate of heat transfer falls for $R_c = 0.1$. From Table II it is ascertained that the increase in source strength maintains almost a constant rate of heat transfer at both the walls for $R = 0.5$.

Acknowledgement

The authors acknowledge their thanks to the referee for the valuable comments and Professor B. P. Acharya for suggestions to improve the paper.

References

1. S. I. Pai, *Viscous Flow Theory*, Vol. I. Van Nostrand, 1956.
2. R. S. Nanda, *J. Phys. Soc. Japan* **13** (1958), 748.
3. M. Katagiri, *J. Phys. Soc. Japan* **17** (1962), 1593.
4. P. K. Muhuri, *J. Phys. Soc. Japan* **18** (1963), 1671.
5. R. S. Rath, S. K. Nayak and B. S. Mohapatra, *Acta Cinencia India*, **1** (1974) 36.
6. S. P. Mishra, *Indian J. pure appl. Phys.* **3** (1965), 354-55.
7. V. M. Soundalgekar, *Rev. Roum Phys.* **16** (1971), 797-804.
8. S. Padhy, Ph. D. Thesis submitted to Utkal University, Bhubaneswar, 1978.
9. J. G. Oldroyd, *Proc. Roy. Soc. Lond.* **A 200** (1950), 523.
10. H. Gieskus, *Rheol. Acta* **9** (1970), 474.

Indian J. pure appl. Math., 20 (3) : 276-283, March, 1988

ON RAYLEIGH WAVES IN GREEN-LINDSAY'S MODEL OF GENERALIZED THERMOELASTIC MEDIA

N. C. DAWN AND S. K. CHAKRABORTY

Department of Mathematics, University of Burdwan, Burdwan 713104

(*Received* 24 *August* 1987; *after revision* 23 *March* 1988)

Rayleigh wave formulation in a half-space exhibiting generalized thermoelasticity properties of Green-Lindsay model has been made. The case of Lord and Shulman's generalized thermoelasticity is shown to occur as a special case. Approximations to the frequency equation have been made for different ranges of the parameters involved. In the case of small coupling, a numerical study has been made to show the effect of the relaxation parameters on the amplitude attenuation factor and also on the velocity of propagation

INTRODUCTION

Thermo-elastic Rayleigh waves in semi-infinite isotropic solids have been studied by Deresiewicz[1], Lockett[2], Chadwick and Sneddon[3], Chadwick[4]. Chakraborty and Pal[5] extended the problem to transversely isotropic medium. These surface waves are found to propagate with a speed in general dependent on the thermal properties as well as on the wave-length of the waves. In the case of small reduced frequency, however, the thermal effect on the wave speed is found to be negligible.

The physical foundations of the above problems were the formulations of Biot[6] and Lessen[7]. The coupled heat conduction equations used by them permitted infinite speeds of propagation of thermal signals. Lord and Shulman[8], proposed generalized thermo-elasticity equations, in which they modified Fourier's heat conduction equation by taking into account the time needed for acceleration of the heat flow. This leads to changes in the energy equation of a coupled theory of theoremo-elasticity by the incorporation of a relaxation term. Green and Lindsay[9] proposed generalized constitutive equations starting from the thermo-dynamical principles without altering Fourier's heat conduction equation. The Green-Lindsay thermoelasticity equations involve two new parameters. Both the Lord-Shulman and Green-Lindsay formulations give finite speeds of thermal wave propagation and also a "second sound effect". However, Green-Lindsay formulation appears to be theoretically more satisfying and Lord-Shulman case follows as a special case of Green-Lindsay in many problems. In the linearized case, according to the Green-Lindsay theory, the relaxation parameters a and a^* in the stress-strain temperature relation and the equation of heat conduction respectively are independent but the inequality $a \geqslant a^* \geqslant 0$ is to be satisfied for

uniqueness of temperature and the speed of heat conduction which depends on a^* is finite only if the stresses depend on the time rate of temperature.

Nayfeh and Nemat-Nasser[11] used the Lord-Shulman theory to study plane thermo-elastic surface waves in a half space. On the basis of Green-Lindsay model, Agarwal[12] studied plane thermo-elastic waves—their propagation and stability. The two results in general are different. They coincide however, in the special case when the two relaxation parameters are equal.

The present paper is concerned with the problem of thermoelastic Rayleigh wave in semi-infinite solids of Green-Lindsay's model. It is found that the relaxation parameters a, a^* for Green-Lindsay theory contribute to the frequency equation terms of order higher than $\chi^{-1/2}$, χ being the reduced frequency. It is seen that when the two relaxation parameters are equal, the results reduce to the case studied by Nayfeh and Nemat-Nasser[11] based on Lord-Shulman's theory.

Approximations to the frequency equations for different ranges of the parameters involved have been obtained. A numerical study of the change of the amplitude attenuation factor and also of velocity of propagation with the relaxation parameter a have been made in the case of small thermo-elastic coupling.

Basic Equations

The equations governing linear thermoelastic interactions in a homogeneous and isotropic solid free from body force and heat sources, as proposed by Green and Lindsay[9] and Green[10] are :

(a) the strain-displacement relations :

$$2e_{ij} = u_{i,j} + u_{j,i} \qquad \text{...(1)}$$

(b) the stress-strain temperature relations :

$$\tau_{ij} = \lambda \, \Delta \, \delta_{ij} + 2 \, \mu \, e_{ij} - \gamma \, (T + a\dot{T}) \, \delta_{ij} \qquad \text{...(2)}$$

(c) the equation of motion :

$$\mu \, \nabla^2 \vec{u} + (\lambda + \mu) \, \nabla \, \text{div} \, \vec{u} - \gamma \, \nabla \, (T + a \, \dot{T}) = \rho \, \ddot{\vec{u}} \qquad \text{...(3)}$$

(d) the equation of heat conduction :

$$\kappa \, \nabla^2 \, T - \rho \, c_v \, (\dot{T} + a^* \, \ddot{T}) - \gamma \, T_0 \, \text{div} \, \dot{\vec{u}} = 0 \qquad \text{...(4)}$$

where

$$a \geqslant a^* \geqslant 0$$

e_{ij} = cartesian components of the linear strain-tensor

τ_{ij} = cartesian components of the linear stress-tensor

δ_{ij} = Kronecker's delta

$\Delta = u_{i,j}$ = dilatation,

$\gamma = (3\lambda + 2\mu)\,\alpha_t$; λ, μ = Lame's constant

α_t = coefficient of linear thermal expansion

κ = coefficient of thermal conductivity,

ρ = constant mass density

c_v = specific heat at constant volume

T = the change in the absolute basic temperature T_0.

a, a^* = thermal relaxation times (constitutive coefficients).

Formulation of the Problem

For Rayleigh type waves in the half space $z \geqslant 0$, using the representation of displacement components :

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z},\; u_y = 0,\; u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \qquad \ldots(5)$$

where ϕ and ψ are functions of x, z and t, eqns. (3) and (4) are satisfied if

$$\frac{\partial^2 \phi}{\partial t^2} = c_1^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{\gamma}{\rho}\,(T + a\,\dot{T}) \qquad \ldots(6)$$

$$\frac{\partial^2 \psi}{\partial t^2} = c_2^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \qquad \ldots(7)$$

and

$$\rho\, c_v\,(T + a^*\,\dot{T}) + \gamma\, T_0\, \frac{\partial}{\partial t} \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \qquad \ldots(8)$$

with $c_1^2 = (\lambda + 2\mu)/\rho,\; c_2^2 = \mu/\rho$.

Using the quantities

$$x' = x/(c_1/\omega^*),\; z' = z/(c_1/\omega^*),\; t' = t\,\omega^*,$$

$$u'_x = \frac{u_x}{(c_1/\omega^*)},\; u'_z = \frac{u_z}{(c_1/\omega^*)},\; T' = \frac{\gamma\, T}{\rho\, c_1^2}$$

$$\phi' = \frac{\phi}{(c_1/\omega^*)^2},\; \psi' = \frac{\psi}{c_1/\omega^*)^2},\; a' = a\omega^*,\; a^{*\prime} = a^*\,\omega^*$$

where $\omega^* = \rho\, c_v\, c_1^2 / \kappa$, in eqns. (6), (7), (8) and suppressing the primes, were obtain the equations in dimensionless form :

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \quad \text{...(9)}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - (T + a\dot{T}) \quad \text{...(10)}$$

$$(\dot{T} + a^* \ddot{T}) + \epsilon \frac{\partial}{\partial t}\left\{\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right)\right\} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \quad \text{...(11)}$$

and

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{v^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \quad \text{...(12)}$$

where the thermo-elastic coupling is given by

$$\epsilon = \frac{\gamma^2 T_0}{\rho^2 c_v c_1^2} \text{ and } v^2 = \frac{c_1^2}{c_2^2}. \quad \text{...(13)}$$

Stress and thermal conditions of the problem on the boundary $z = 0$ are $\tau_{zx} = \tau_{zz} = 0$ and $\frac{\partial T}{\partial z} + hT = 0$. ...(14)

Solution

For thermoelastic surface waves in the half-space propagating in x-direction the functions $\{T, \phi, \psi\}$ may be taken in the form

$$\{T, \phi, \psi\} = \{\hat{T}(z), \hat{\phi}(z), \hat{\psi}(z)\} \exp i(\eta x - \chi t). \quad \text{...(15)}$$

Substituting (15) in (9) – (11) and remembering that $\hat{T}, \hat{\phi}, \hat{\psi} \to 0$ as $z \to \infty$ for surface waves, the solution is obtained as

$$\phi = [A \exp(-\eta \beta_1 z) + B \exp(-\eta \beta_2 z)]. \exp i(\eta x - \chi t) \quad \text{...(16)}$$

$$\psi = C. \exp[-\eta \beta_3 z + i(\eta x - \chi t)] \quad \text{...(17)}$$

and

$$T = \frac{1}{1 - ia\chi} [A\{\chi^2 + \eta^2(\beta_1^2 - 1)\} \exp(-\eta \beta_1 z)$$

$$+ B\{\chi^2 + \eta^2(\beta_2^2 - 1)\} \exp(-\eta \beta_2 z)]. \exp i(\eta x - \chi t) \quad \text{...(18)}$$

where $\beta_3^2 = (1 - c^2 v^2)$, $c^2 = x^2/\eta^2$ and β_1, β_2 are those roots of the following equation for which Re $(\beta) > 0$:

$$\beta^4 + \beta^2[-2 + c^2\{(1 + a^* + \epsilon a) + i(1 + \epsilon)/\chi\}] + 1 - c^2 \{(1 + a^*$$

$$+ \epsilon a) + \frac{i}{x}(1 + \epsilon)\} + c^4(a^* + \frac{i}{x}) = 0. \quad \text{...(19)}$$

On using the boundary conditions (14), $A : B : C$ are eliminated giving rise to the frequency equation

$$(2 - \nu^2 c^2)^2 (\beta_1^2 + \beta_2^2 + \beta_1 \beta_2 - 1 + c^2) - 4 \beta_1 \beta_2 \beta_3 (\beta_1 + \beta_2)$$

$$= - \frac{h}{\eta} \{(\beta_1 + \beta_2)(2 - \nu^2 c^2)^2 - 4 \beta_3 (\beta_1 \beta_2 + 1 - c^2)\}. \quad \ldots(20)$$

Discussion

Equation (20) along with (19) determines the velocity of thermoelastic Rayleigh waves in a relaxing medium. In order to have an idea of the effect of the relaxation parameters on the velocity of propagation we shall study below in detail the special case of small thermoelastic coupling :

Case (a): Small Thermal Coupling

For most of the materials ϵ is small at normal temperature. Hence we may make an approximation of the frequency equation assuming $\epsilon \ll 1$.

For $\epsilon \ll 1$, we get from (19)

$$\beta_1 \simeq (1 - c^2)^{1/2} \left[1 - \frac{\epsilon}{2} \frac{c^2\left(a + \frac{i}{\chi}\right)}{(1 - a^* - \frac{i}{\chi})(1 - c^2)} \right. \quad \ldots(21)$$

$$\beta_2 \simeq \{1 - c^2 (a^* + \frac{i}{\chi})\}^{1/2} [1 + \frac{\epsilon}{2} \frac{c^2 (a^* + \frac{i}{x})(a + \frac{i}{x})}{(1 - a^* - i/\chi) \{1 - c^2 (a^* + l/\chi}] .$$

The frequency equation (20) with $h = 0$ thus reduces to an equation in which the parameters a, a^* are involved :

$$(2 - \nu^2 c^2)^2 [1 - c^2 \{(a^* + \epsilon a) + \frac{i}{\chi}(1 + \epsilon)\} + (1 - c^2)^{1/2} \{1 - c^2 (a^*$$

$$+ \frac{i}{\chi})\}^{1/2} (1 - \frac{\epsilon}{2} \frac{c^2 (a + i/\chi)}{(1-c^2) \{1-c^2 (a^* + i/\chi)\}})] - 4 (1 - c^2 \nu^2)^{1/2}$$

$$(1-c^2)^{1/2} \{1 - c^2 (a^* + \frac{i}{x})\}^{1/2} [1 - \frac{\epsilon}{2}] \frac{c^2 (a + i/\chi)}{(1 - c^2) \{1 - c^2 (a^* + i/\chi)\}]}$$

$$[(1-c^2)^{1/2} \{1 - \epsilon/2 \frac{c^2 (a + i/\chi)}{(1 - a^* - i/\chi)(1 - c^2)}\} + \{1 - c^2 (a^* + i/\chi)\}^{1/2}$$

$$\{1 + \epsilon/2 \frac{c^2 a^* + i/\chi)(a + i/\chi)}{(1 - a^* - i^\prime\chi) \{1 - c^2 (a^* + i/\chi)\}}\}] = 0. \quad \ldots(22)$$

If we put

$$c^2 = c^{*2} + \epsilon\,(\xi_1 + i\xi_2)$$

where c^* is the classical Rayleigh wave velocity and ξ_1 and ξ_2 are two reals depending on the reduced frequency χ and a, a^*,

then

$$\eta = \frac{\chi}{c^*}\,(1 - \frac{\epsilon\,\xi_1}{2c^{*2}} - \frac{i\,\epsilon\,\xi_2}{2\,c^{*2}})$$

The velocity of propagation being $(c^* + \frac{\epsilon\,\xi_1}{2\,c^*})$ the waves exhibit dispersion and the amplitude-attenuation factor $= \exp\left[\frac{\epsilon\,\chi\,\xi_2\,x}{2c^{*3}}\right]$ with $\xi_2 < 0$.

Substituting the above value of c^2 in (22) and solving numerically for different values of the parameters we have calculated the velocity of propagation and the amplitude-attenuation factor. In Fig. 1, we have plotted velocity of propagation $(\chi)\,10^3$ against the relaxation parameter a for $\epsilon = 0.05$, $(\chi) = 0.1$ and $a^* = 0.2$ and in Fig. 2 amplitude attenuation factor $(\chi)\,10^3$ has been plotted against the relaxation parameter a for the same value of ϵ, χ and a^* for the case $x = 1$.

From Fig. 1, it is seen that the velocity of propagation decreases gradually as the relaxation parameter a increases, though the change is very small. This may be interpreted as solving down of Rayleigh waves with increasing heat conductivity (Nayfeh and Nemat-Nasser[11], p. 55).

Similarly, in the Fig. 2, it is seen that amplitude-attenuation factor decreases with increase in the relaxation parameter a.

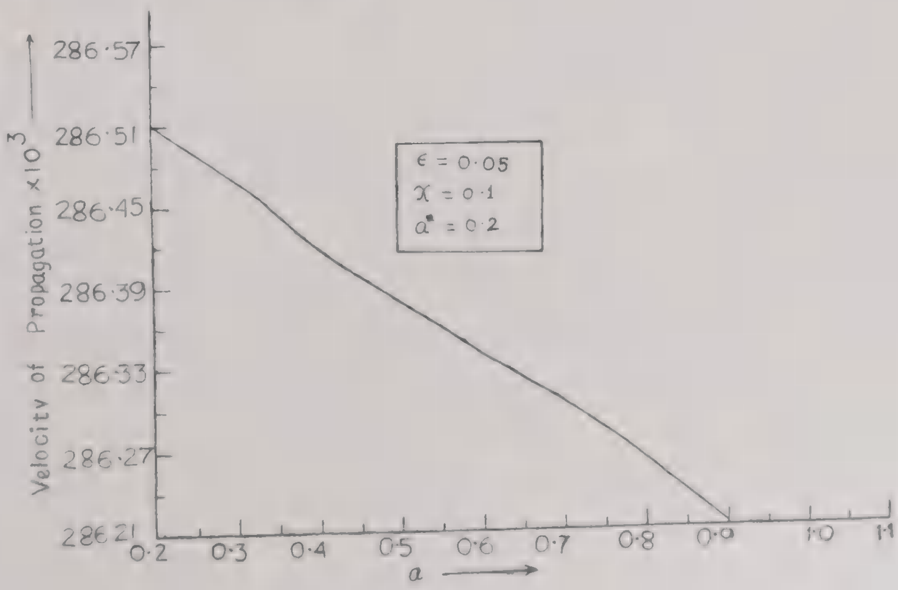

FIG. 1.

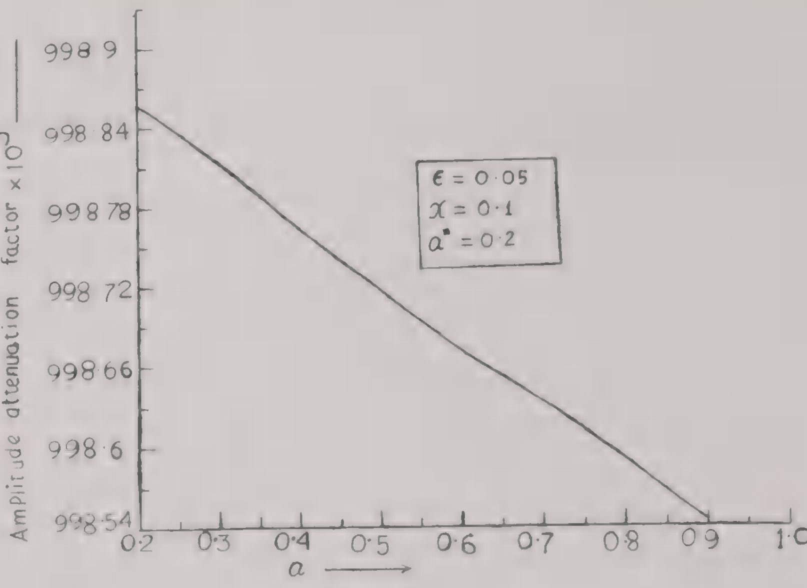

FIG. 2.

Case (b) : Small reduced frequency $\chi \ll 1$.

It is seen that the characteristic frequency ω^* is much greater than the frequencies attainable even in experiments employing ultrasonic pulses. Thus we may assume $\omega \ll \omega^*$, i. e. $\chi \ll 1$. Expanding the terms of the frequency equation in powers of the reduced frequency χ, the frequency equation (20) can be written as

$$(2 - \nu^2 c^2)^2 \Big[- \frac{i c^2}{\chi} (1 + \epsilon) + \frac{(1 - i)\, c}{\sqrt{2}.\, \chi^{1/2}} (1 + \epsilon - c)^{1/2} + 1 - c^2 (a^* + \epsilon a) + \frac{(1 + i)}{2\sqrt{2}} \frac{\{1 - c^2 (1 + a^* + \epsilon a) + a^* c^4\}}{c\,(1 + \epsilon - c^2)^{1/2}} .\, \chi^{1/2} \Big]$$

$$- 4\,(1 - \nu^2 c^2)^{1/2} \Big[- \frac{i c^2}{\chi} (1 + \epsilon)^{1/2} (1 + \epsilon - c^2)^{1/2} + \frac{(1 + i)}{\sqrt{2}} \frac{c}{\chi^{1/2}} \cdot \frac{(1 + \epsilon - c^2)}{(1 + \epsilon)^{1/2}} + (1 + \epsilon)$$

$$\times \frac{\{3 - 2c^2 (1 + a^* + \epsilon a) + a^* c^4\} - c^2 \{2 - c^2 (1 + a^* + \epsilon a)\}}{2\,(1 + \epsilon)^{1/2} (1 + \epsilon - c^2)^{1/2}}$$

$$+ \frac{(1 + i)}{2\sqrt{2}} \cdot \frac{\chi^{1/2}}{c} \frac{\{1 - c^2 (1 + a^* + \epsilon a) + a^* c^4\}}{(1 + \epsilon)^{1/2}} \Big]$$

$$= - \frac{h}{\eta} \Big[\Big\{ \frac{1 - i}{\sqrt{2}} \cdot \frac{c}{\chi^{1/2}} \cdot (1 + \epsilon)^{1/2} + \frac{(1 + \epsilon - c^2)^{1/2}}{(1 + \epsilon)^{1/2}} + \frac{(1 + i)}{2\sqrt{2}}$$

$$\times \frac{\chi^{1/2}}{c} \cdot \frac{2 - c^2 (1 + a^* + \epsilon\, a)}{(1 + \epsilon)^{1/2}} \Big\} (2 - \nu^2 c^2)^2 - 4\,(1 - \nu^2 c^2)^{1/2}$$

(*equation continued on p. 283*)

$$\times \left\{ \frac{1-i}{\sqrt{2}} \cdot \frac{c}{\chi^{1/2}} \cdot (1 + \epsilon - c^2)^{1/2} + \frac{1+i}{2\sqrt{2}} \frac{\chi^{1/2}}{c} \right.$$

$$\times \frac{1 - c^2 (1 + a^* + \epsilon a) + a^* c^4}{(1 - \epsilon - c^2)^{1/2}} + 1 - c^2\}]. \qquad ...(24)$$

From (24), we therefore find that the relaxation parameter's 'a', a^* are involved only in the terms of order higher than $\chi^{-1/2}$. Hence the effect of these parameters is negligible for small χ.

The results corresponding to zero flux of heat across the boundary may be obtained by putting $h = 0$. Therefore putting $h = 0$ in the frequency equation (24) and retaining terms only $O(\chi^{-1})$ we get

$$(2 - \nu^2 c^2)^2 = 4 (1 - \nu^2 c^2)^{1/2}. \frac{1 - \epsilon - c^2}{(1 + \epsilon)^{1/2}}. \qquad ...(25)$$

Equation (25), is identital with the corresponding equation in coupled thermo-elastic medium[2].

Case (c): Lord-Shulman Case

It is to be noted that eliminating the temperature from the displacement equation (3) and the heat conduction equation (4) and then putting $a = a^*$ results in an equation for displacement which is the same as in Lord-Shulman's case with 'a' as the relaxation parameter. Therefore in Green-Lindsay's theory, all wave problems and source problems reduce to Lord-Shulman's case with $a = a^*$ playing the role of relaxation parameter.

If we put $a = a^*$ then the characteristic equation (19) and the frequency equation (20) coincide with the case corresponding to Lord-Shulman theory[11].

References

1. H. Deresiewicz, *J. Acoust. Soc. Am.* **29** (1957) 204-209.
2. F. J. Lockett, *J. Mech. Phys. Solids.* **7** (1958), 71-75.
3. P. Chadwick and I. N. Senddon, *J. Mech. Phys. Solids* **6** (1958), 223-30.
4. P. Chadwick, *Thermo-elasticity. The Dynamical Theory. Progress in Solid Mechnnics*, Vol. 1, Chapter 6. North-Holland Publishing Co. Amsterdam 1960.
5. S. K. Chakraborty and R. P. Pal, *Pure Appl. Geophy.* **76** (1969), 79-86.
6. M. A. Biot, *J. Appl. Phys.* **27** (1956), 240-53.
7. M. Lessen, *J. Mech. Phys. Solids* **5** (1956), 57-61.
7. H. W. Lord and Y. Shulman, *J. Mech. Phys. Solids* **15** (1967), 299-309.
9. A. E. Green and K. A. Lindsay, *J. Elasticity* 2 (1972), 1-7.
10. A. E. Green, *Mathematica* **19** (1972), 69-75.
11. A. Nayfeh and S. Nemat-Nasser, *Acta Mech.* **12** (1971) 53-69.
12. V. K. Agarwal, *Acta Mech.* **31** (1979), 185-98.

Indian J. pure appl. Math., 20 (3) : 284-291, March 1989

ROCHE HARMONICS FOR STELLAR MODELS DISTORTED BY DIFFERENTIAL ROTATION

V. P. SINGH AND VANDANA GUPTA
Institute of Paper Technology, University of Roorkee, Saharanpur 247001

(*Received* 22 *March* 1988)

Kopal[1,2] obtained expliciting expressions for Roche harmonics associated with the stellar models of a star distorted by rotation (solid body) forces Following Kopal's approach we obtain explicit expressions for Roche harmonics associated with the stellar model of a star distorted by the differential rotation (the law of differential rotation has been assumed of the form $\omega = b_1 + b_2 s^2$, where ω is the angular velocity of rotation of a fluid element distant s from the axis of rotation and b_1, b_2 are certain constants).

1. INTRODUCTION

Kopal[2] introduced a family of new auxiliary functions, which he called the Roche harmonics, generated by the solution of the Laplace equation

$$\nabla^2\phi = 0 \quad \ldots(1)$$

derived in terms of the Roche curvilinear coordinates, which are associated with the Roche equipotential surfaces in the same way as the spherical harmonics are associated with a sphere. Roach[3] discussed some more theoretical aspects of Roche harmonics. In fact, as was pointed out by Kopal (*op. cit.*), spherical harmonics may be regarded as a limiting case of the more general Roche harmonics, and the latter may be more appropriate for the study of many problems arising in double star astronomy. Kopal obtained explicit expressions for Roche harmonics for rotationally or tidally distorted stellar models. In the case of Roche harmonics associated with equipotential surfaces which are being distorted by the rotational forces, Kopal obtained explicit expressions by taking into account the effects of solid body rotation only. Singh[4], Mohan, Singh[5] extended the analysis of Roche coordinates taking into account the effect of differential rotation. The law of differential rotation has been assumed to be of the form $\omega = b_1 + b_2 s^2$, where ω is the angular velocity of an element distant s from the axis of rotation, b_1 and b_2 are constants.

In this paper we present the analysis of Roche harmonics for stellar models distorted by differential rotation, using the above cited law of differential rotation as $\omega = b_1 + b_2 s^2$, to obtain the explicit expressions of Roche harmonics associated with Roche equipotential surfaces distorted by differential rotation. This is performed in Section 4.

For completeness the work of Kopal[1,2] is respected in Section 2 considering solid body rotation effects only. The system of Roche coordinates for the Roche-model distorted by differential rotation as obtaine dby Mohan and Singh[5] are presented in section 3. Concluding remarks are given in section 5.

2. Explicit Expressions of Roche Harmonics for Roche Equipotential Surfaces Distorted solid by Body Rotation Forces

Following Kopal[1] it can be shown that,

$$\xi = \frac{1}{r} + nr^2 (1 - \nu^2) = \text{constant} \qquad ...(2)$$

represents the equipotential surface of a star distorted by rotation (solid body) forces. Here $n = \omega^2/2$, ω being the angular velocity of rotation (solid body rotation), r is the dimensionless measurement of distance and $\nu = Z/r$ is the direction cosine with respect to Z-axis, the axis of rotation. The surfaces generated by setting ξ = constant in eqn. (2) are referred to as Roche equipotentials.

Choosing ξ as defined in eqn. (2) as one coordinate Kopal has shown that the second Roche coordinate η is given by

$$\eta = \phi. \qquad (3)$$

The expression for the third Roche coordinate ζ is not possible in closed analytic form. Kopal obtained expression for this third Roche coordinate ζ in ascending powers of the angular velocity of rotation as

$$\cos \zeta = \sum_{j=0}^{\infty} (2n)^j . r^{3j} X_j (\nu) \qquad ...(4)$$

where $X_0 (\nu) = 1$, $X_1 (\nu) = - \frac{1}{3} (1 - \nu^2)$, while for $j > 1$ all subsequent $X_j (\nu)$'s can be generated with the help of the recursion formula

$$3j X_j + (1 - \nu^2) [(\nu X_{j-1})' - 3 (j - 1) X_{j-1}] = 0.$$

Here prime denotes differentiation with respect to ν.

Kopal also obtained expressions for the metric coefficients h_1, h_2 and h_3 in terms of polar spherical coordinates upto second order terms in square of angular velocity of solid body rotation.

These are :

$$h_1 (\xi, \zeta) = r_0^2 [1 + 4n r_0^3 \sin^2\zeta - \frac{1}{3} n^2 r_0^6 \sin^2\zeta (22 - 85 \sin^2\zeta) + ...]$$

$$h_2 (\xi, \zeta) = r_0 \sin\zeta [1 - \frac{1}{3} nr_0^3 (2 - 5 \sin^2\zeta) + \frac{1}{9} n^2 r_0^6 (2 - 50 \sin^2\zeta + 7 \sin^4\zeta) + ...;$$

$$h_3(\xi, \zeta) = r_0 [1 - \tfrac{1}{3} n r_0^3 (2 - 7 \sin^2\zeta) + \tfrac{1}{9} n^2 r_0^6 (2 - 88 \sin^2 \zeta + 145 \sin^4\zeta) + \ldots]. \quad \ldots(5)$$

Here $r_0 = 1/\xi$ denotes the mean fractional radius of the equipotential surface $\xi =$ constant.

Making use of h_1, h_2 and h_3 as given by (5), Kopal[2] has shown that the Laplace equation (1) becomes

$$\nabla^2 \phi = [1 - nr_0^3 (1 - \nu_0^2) + \frac{2}{3} n^2 r_0^6 (1 - \nu_0^2)(9 + 13\nu_0^2)]$$

$$\frac{\partial}{\partial r_0}\left(r_0^2 \frac{\partial \phi}{\partial r_0}\right) - 4n r_0^4 \frac{\partial \phi}{\partial r_0} + \left[1 - \frac{2}{3} nr_0^3 (5 - 7\nu_0^2)\right.$$

$$\left. - \frac{1}{9} n^2 r_0^6 (43 - 194 \nu_0^2 + 143 \nu_0^4\right] (1 - \nu_0^2) \frac{\partial^2 \phi}{\partial \nu_0^2}$$

$$- \left[1 + \frac{4}{3} nr_0^3 \nu_0^2 + \frac{1}{9} n^2 r_0^6 (75 - 34 \nu_0^2 - 33 \nu_0^4\right]$$

$$2\nu_0 \frac{\partial \phi}{\partial \nu_0} = 0 \quad \ldots(6)$$

where

$$\nu = \cos \zeta, \ r_0 = 1/\xi.$$

Assuming a series of the form

$$\phi = \sum_j a_j r_0^j R_j \quad \ldots(7)$$

where

$$R_j = P_j(\nu_0) + nr_0^3 X_2^{(j)}(\nu_0) + n^2 r_0^6 X_4^{(j)}(\nu_0) + \ldots. \quad \ldots(8)$$

Kopal has shown that for $j > 1$, $X_2^{(j)}(\nu_0)$ and $X_4^{(j)}(\nu_0)$ are found to be of the form

$$X_2^{(2)}(\nu_0) = -(1 - \nu_0^2)(1 - 5\nu_0^2) \quad \ldots(9)$$

$$X_2^{(3)}(\nu_0) = -\tfrac{1}{2}(1 - \nu_0^2)(11 - 25 \nu_0^2) \nu_0 \quad \ldots(10)$$

$$X_2^{(4)}(\nu_0) = \frac{1}{6}(1 - \nu_0^2)(9 - 120 \nu_0^2 + 175 \nu_0^4) \quad \ldots(11)$$

$$X_2^{(5)}(\nu_0) = \frac{5}{8}(1 - \nu_0^2)(17 - 98\nu_0^2 + 105\nu_0^4) \nu_0 \quad \ldots(12)$$

$$X_4^{(2)}(\nu_0) = -\frac{1}{6}(1 - \nu_0^2)(21 - 172\nu_0^2 + 175 \nu_0^4) \quad \ldots(13)$$

etc.

Equation (8) with (9)-(13) constitute the explicit form of the Roche harmonics associated with Roche equipotential surfaces (2) distorted by solid body rotation forces.

3. Roche Coordinates for Roche Model Distorted by Differential Rotation

For the Roche model of mass M, rotating according to the law

$$\omega = b_1 + b_2 s^2 \quad \text{.. (14)}$$

the equation of hydrostatic equilibrium may be written in the form

$$d\Omega = dv + \tfrac{1}{2} \omega^2 d (s^2) \quad \text{...(15)}$$

where Ω denotes potential at a point P distant r from the centre of the star. $v = GM/r$, is the gravitational potential, ω is the angular velocity of rotation of an element of the fluid distant s from the axis of rotation.

On substituting (14) in (15) we have

$$d\Omega = dv + \tfrac{1}{2} (b_1^2 + 2b_1 b_2 s^2 + b_2^2 s^4) d (s^2).$$

On integration and simplification this gives

$$\Omega = \frac{GM}{r} + \tfrac{1}{2} (x^2 + y^2) [b_1^2 + b_1 b_2 (x^2 + y^2) + \tfrac{1}{3} b_2^2 (x^2 + y^2)^2] \quad \text{....(16)}$$

In spherical polar coordinates

$$\left.\begin{aligned} x &= r \cos \phi \sin \theta = r\lambda \\ y &= r \sin \phi \sin \theta = r\mu \\ z &= r \cos \theta \qquad\quad = r\nu \end{aligned}\right\} \quad \text{...(17)}$$

Equation (16) may be expressed in non-dimensional form as

$$\xi = \frac{1}{r} + \tfrac{1}{2} r^2 (1 - \nu^2) [b_1^2 + b_1 b_2 (1 - \nu^2) + \tfrac{1}{3} b_2^2 r^4 (1 - \nu^2)^2] \quad \text{...(18)}$$

where $\xi = R\Omega/GM$ is a non-dimensional parameter denoting potential and ω is now non-dimensional angular velocity in units of GM/R^3, R being the unit of distance.

The surfaces generated by setting ξ = constant in (18) are referred as the Roche equipotentials.

Now if we take $r_0 = 1/\xi$ as our first approximation to the distance of the equipotential surface from the centre, Mohan and Singh[5] have shown that

$$r = r_0 [1 + \tfrac{1}{6} r_0^3 (1 - \nu_0^2) (b_1^2 + b_1 \omega_0 + \omega_0^2)] \quad \text{...(19)}$$

where

$$\nu_0 = \cos\zeta \text{ and } \omega_0 = b_1 + b_2 r_0^2 (1 - \nu_0^2).$$

In the system of Roche coordinates (ξ,η,ζ), the ξ-coordinate is defined by Roche equipotential surfaces of the form (18) while coordinates η and ζ are defined by the requirement that they are orthogonal to ξ as well as to each other. In this triple orthogonal system of Roche coordinates, Mohan and Singh[5] have shown that the second and third coordinates are given by

$$\eta = \frac{\lambda}{\sqrt{1-\nu^2}} \qquad \ldots(20)$$

and

$$\cos\zeta = \nu\left[1 - \frac{(1-\nu^2)\, r^3}{105}\,(15\omega^2 + 12 b_1\,\omega + 8 b_1^2) + \ldots\right] \qquad \ldots(21)$$

whereas the expression for η given in (20) is exact, the expression for ζ obtained in (21) is correct upto second order terms in angular velocity ω.

Mohan and Singh[5] also obtained the explicit expressions for the metric coefficients h_1, h_2 and h_3 correct upto sectond order terms in ω, which are found to be

$$h_1(\xi,\zeta) = r_0^2\left[1 + \frac{r_0^3 \sin^2\zeta}{3}(4\omega_0^2 + b_1\,\omega_0 + b_1^2) + \ldots\right]$$

$$h_2(\xi,\zeta) = r_0 \sin\zeta\left[1 - \frac{r_0^3}{210}(15\,\omega_0^2 + 12 b_1\,\omega_0 + 8 b_1^2) + \frac{r_0^3 \sin^2\zeta}{210}(65\omega_0^2 + 9 b_1\,\omega_0 + 51 b_1^2) + \ldots\right]$$

$$h_3(\xi,\zeta) = r_0\left[1 - \frac{r_0^3}{210}(150\,\omega_0^2 - 48\, b_1\,\omega_0 - 32 b_1^2) - \sin^2\zeta\,(215\omega_0^2 + 11 b_1\,\omega_0 + 19 b_1^2) + \ldots\right] \qquad \ldots(22)$$

4. Explicit Expressions of Roche Harmonics for Roche Equipotential Surfaces Distorted by Differential Rotation Forces

Using expressions of h_1, h_2 and h_3 as given by (22), the Laplacian

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial\zeta}\right)\right]$$

now becomes

$$\nabla^2 = \left[1 - \frac{r_0^3}{105}(1-\nu_0^2)(280\omega_0^2 + 70\, b_1\,\omega_0 + 70\, b_1^2)\right]\frac{\partial}{\partial r_0}\left(r_0^2\frac{\partial}{\partial r_0}\right)$$

(*equation continued on p.* 289)

$$- \left[\frac{3r_0^2}{105} \ (90\omega_0^2 - 12b_1 \omega_0 - 8b_1^2) \right.$$

$$\left. + \frac{2r_0^4 b_2}{105} (1 - \nu_0^2)(180\omega_0 - 12b_1) \right] r_0^2 \frac{\partial}{\partial r_0}$$

$$- \left[2 + \frac{r_0^8}{210} (600\omega_0^2 - 192 \ b_1\omega_0 - 128 \ b_1^2) \nu_0^2 \right.$$

$$+ (1 - \nu_0^2) \ b_2 \ r_0^2 \ ((520 \ (1 - \nu_0^2)$$

$$\left. + 480) \ \omega_0 + (236 \ (1 - \nu_0^2) - 144) \ b_1) \right] \nu_0 \frac{\partial}{\partial \nu_0}$$

$$+ \left[1 - \frac{r_0^8}{210} - \nu_0^2 \ (430\omega_0^2 + 22 \ b_1 \ \omega_0 + 38b_1^2) \right.$$

$$\left. + 130 \ \omega_0^2 + 118b_1 \ \omega_0 + 102 \ b_1^2 \right] (1 - \nu_0^2) \frac{\partial^2}{\partial \nu_0^2} \ .$$

Therefore the Laplace equation (1) takes the explicit form in this case as

$$\left[1 - \frac{r_0^8}{105} \ (1 - \nu_0^2)(280 \ \omega_0^2 + 70b_1 \ \omega_0 + 70b_1^2) \right] \frac{\partial}{\partial r_0} \left(r_0^2 \ \frac{\partial \phi}{\partial r_0} \right)$$

$$- \left[\frac{3r_0^2}{105} (90\omega_0^2 - 12b_1 \ \omega_0 - 8b_1^2) + \frac{2r_0^4 \ b_2}{105} \ (1 - \nu_0^2) \right.$$

$$\left. (180\omega_0 - 12b_1) \right] r_0^2 \ \frac{\partial \phi}{\partial r_0}$$

$$- \left[2 + \frac{r_0^8}{210} (600 \ \omega_0^2 - 192 \ b_1 \ \omega_0 - 128b_1^2) \ \nu_0^2 \right.$$

$$+ (1 - \nu_0^2) \ b_2 \ r_0^2 \ ((520 \ (1 - \nu_0^2)$$

$$\left. + 480) \ \omega_0 + 236 \ (1 - \nu_0^2) - 144) \ b_1) \right] \nu_0 \ \frac{\partial \phi}{\partial \nu_0}$$

$$+ \left[1 - \frac{r_0^8}{210} - \nu_0^2 \ (430\omega_0^2 + 22b_1 \ \omega_0 + 38b_1^2) \right.$$

$$\left. + 130 \ \omega_0^2 + 118b_1 \ \omega_0 + 102b_1^2 \right] (1 - \nu_0^2) \ \frac{\partial^2 \phi}{\partial \nu_0^2} = 0. \qquad \text{.. (23)}$$

As in section 2, the solution of this equation can be obtained in series form where

$$\phi = \sum_j a_j \ r_0^j \ R_j$$

and setting

$$R_j = P_j(\nu_0) + \frac{\omega_0^2}{2} \; r_0^3 \; X_2^{(j)}(\nu_0) + \ldots \qquad \ldots(24)$$

It can be shown (consistent with the adopted scheme of approximation) that the functions $X_2^{(j)}(\nu_0)$ assume the forms because of the influence of the terms pertaining b_2^2 and $b_1 b_2$, as given by

$$X_2^{(2)}(\nu_0) = -1.8571\,\nu_0^4 + 2.1905\,\nu_0^2 - .3333 \qquad \ldots(25)$$

$$X_2^{(3)}(\nu_0) = -4.6428\,\nu_0^5 + 6.5714\,\nu_0^3 - 1.9286\,\nu_0 \qquad \ldots(26)$$

$$X_2^{(4)}(\nu_0) = -10.8333\,\nu_0^6 + 17.9762\,\nu_0^4 - 7.6428\,\nu_0^2 + .5 \qquad \ldots(27)$$

$$X_2^{(5)}(\nu_0) = -24.375\,\nu_0^7 + 46.458\,\nu_0^5 - 25.7440\,\nu_0^3 + 3.6607\,\nu_0; \qquad \ldots(28)$$

$$X_2^{(2)}(\nu_0) = -2.7\,\nu_0^4 + 3.2\,\nu_0^2 - .5 \qquad \ldots(29)$$

$$X_2^{(3)}(\nu_0) = -6.75\,\nu_0^5 + 9.6\,\nu_0^3 - 2.85\,\nu_0 \qquad \ldots(30)$$

$$X_2^{(4)}(\nu_0) = -15.75\,\nu_0^6 + 26.25\,\nu_0^4 - 11.25\,\nu_0^2 + .75 \qquad \ldots(31)$$

$$X_2^{(5)}(\nu_0) = -.33.4375\,\nu_0^7 + 67.8125\,\nu_0^5 - 37.8125\,\nu_0^3 + 5.4375\,\nu_0 \ldots(32)$$

etc.

Equation (24) with (25) - (32) constitutes the explicit form of the Roche harmonics associated with the Roche equipotentials surfaces (18) up to second order approximation of the differential rotation terms, which indeed, is the outcome of our analysis.

5. Concluding Remarks

The influence of b_1^2 terms on $X_2^{(j)}(\nu_0)$ is same as obtained by Kopal[2]. In the present analysis we could only present the results which are because of the influence of the terms pertaining b_2^2 and $b_1 b_2$. This is because of our approximation scheme for angular velocity ω upto second order. The consideration of higher order terms in ω, should lead to the appropriate formulation of this problem which we intend to investigate this problem in subsequent study.

It may be pointed out that although we have studied here the problem of Roche harmonics associated with equipotential surfaces by assuming the Roche model of the

star, the present method of Roche coordinates can be used also when some more realistic structure is assumed for the interior of the model. We can still either approximate the distorted equipotentials of such stars by Roche model or use their more realistic forms by using the system of Clairaut's coordinates (cf. Kopal[6,7]), when the law of differential rotation may be assumed of the form $\omega = b_1 + b_2 s^2$.

ACKNOWLEDGEMENT

The authors will like to express their gratitude to Professor N. J. Rao, Director, Institute of Paper Technology (University of Roorkee), Saharanpur, for providing necessary research facilities. They are also thankful to Professor C. Mohan, Department of Mathematics, University of Roorkee, for valuable discussions and suggestions. The research is supported in part by DRIL grant No. 106/99 (University of Roorkee).

REFERENCES

1. Z. Kopal, *Astrophys. Space Sci.* 8 (1970), 149–171.
2. Z Kopal, *Adv. Astr. Astrophys.* 9 (1972), 1–9.
3. G. F. Roach, *Astrophys. space, Space Sci.* 36 (1975), 159–67.
4. V. P. Singh, Ph. D. thesis Ch. IV University of Roorkee, India, 1978.
5. C. Mohan and V. P. Singh, *Proc. Nat. Aca. Sci., India,* 50 A (1980), 25–31.
6. Z. Kopal, *Astrophys. Space Sci.* 70 (1980), 407.
7. Z. Kopal, *Astrophys. Space Sci.* 76 (1981), 187–212.

Indian J. pure appl. Math., 20 (3) : 292–295 March, 1989

INCOMING WATER WAVES AGAINST A VERTICAL CLIFF IN A TWO-FLUID MEDIUM

P. K. KUNDU

Department of Applied Mathematics, Calcutta University, 92 *A. P. C. Road Calcutta* 700009

(*Received* 23 *December* 1987; *after revision* 22 *July* 1988)

The two-dimensional problem of incoming waves against a vertical cliff in a two-fluid medium is considered in this paper. The velocity potentials describing the motion in each of the fluids are obtained. Known results for the one-fluid medium are recovered when the density of the upper fluid is made to vanish.

1. INTRODUCTION

Stoker[2,3] considered the problem of incoming surface water waves in a deep ocean bounded on one side by a vertical cliff. He solved the two-dimensional problem by employing a technique based on the complex variable theory and the three-dimensional problem by a reduction procedure. No reflection of waves by the cliff is assumed so that a source/sink type behaviour of the potential function there is necessary to account for this and as such the wave amplitude is required to be logarithmically infinite at the shore line. However, if the effect of surface tension at the free-surface is taken into account, this is not necesary, so that the potential remains regular there. Packham[1] solved the corresponding two-dimensional problem by using a technique based on the Fourier sine transform.

The present paper is concerned with the two-dimensional problem of incoming internal waves at the interface of a two-fluid medium against a vertical cliff in the presence of interfacial tension thereby generalising Packham's[1] problem. The velocity potentials in both the fluids are obtained in a straight forward manner.

2. STATEMENT AND SOLUTION OF THE PROBLEM

We consider the irrotational motion of two inviscid incompressible fluids of infinite horizontal extent towards the right and bounded by a vertical cliff on the left and of densities ρ_1 and ρ_2 ($< \rho_1$) respectively under the action of gravity only. The lower fluid extends infinitely downwards while the upper extends infinitely upwards. We choose a rectangular cartesian co-ordinate system such that the y-axis points vertically downwards into the lower fluid, then the cliff is the plane $x = 0$ and $y = 0$, $x > 0$ is

the position of the undisturbed interface. The velocity potentials $\Phi_1(x, y, t)$ and $\Phi_2(x, y, t)$ for the lower and upper fluids respectively, under assumption of time-harmonic motion, can be described by

$$\Phi_j(x, y, t) = \text{Re}\,[\varphi_j(x, y \exp)(-i\sigma t)]\ (j = 1, 2)$$

where σ is the circular frequency. Then φ_j's satisfy

$$\nabla^2 \varphi_j = 0 \text{ in the respective region} \qquad \ldots(2.1)$$

$$\left.\begin{array}{l} \varphi_{1y} = \varphi_{2y} \\ K\varphi_1 + \varphi_{1y} - s(K\varphi_2 + \varphi_{2y}) + M \left\{ \begin{array}{l} \varphi_{1yyy} \\ \varphi_{2yyy} \end{array} \right. = 0 \text{ on } y = 0,\ x > 0 \end{array}\right\} \qquad \ldots(2.2)$$

$$\varphi_{jx} = 0 \text{ on } x = 0\ (y > 0 \text{ for } j = 1 \text{ and } y < 0 \text{ for } j = 2) \qquad \ldots(2.3)$$

$$\varphi_j\text{'s remain finite at the origin} \qquad \ldots(2.4)$$

$$\nabla \varphi_j \to 0 \text{ as } y \to \pm \infty \qquad \ldots(2.5)$$

(the upper sign is for $j = 1$ and the lower sign for $j = 2$) and finally

$$\varphi_j \sim \pm \exp(\mp k_0 y - ik_0 x) \text{ as } x \to \infty \qquad \ldots(2.6)$$

where k_0 is the unique positive real root of

$$k(1 + M' k^2) - L = 0 \qquad \ldots(2.7)$$

$$M' = M/(1 - s),\ L = K(1 + s)/(1 - s),\ K = \sigma^2/g$$

$$s = \rho_2/\rho_1,\ M = T/(\rho_1 g).$$

T being the interfacial tension and g the gravity.

Here, (2.1) is the equation of continuity in either of the fluids, (2.2) is the linearized kinematical and dynamical conditions at the interface, (2.3) is the condition at the cliff, (2.4) is the condition at the shore-line, (2.5) is the condition of no motion at infinite depth and height, and (2.6) is due to the incoming nature of the waves as $x \to \infty$ moving towards the cliff.

To solve for $\varphi_j(x, y)$ we write

$$\varphi_j(x, y) = \pm 2 \exp(\mp k_0 y) \cos(k_0 x) + \psi_j(x, y). \qquad \ldots(2.8)$$

Then ψ_j's satisfy the same eqns (2.1) to (2.5) as satisfied by φ_j's and

$$\psi_j(x, y) \to \mp \exp(\mp k_0 y + i k_0 x) \text{ as } x \to \infty. \qquad \ldots(2.9)$$

(2.9) states that ψ_j's behave as outgoing waves as $x \to \infty$.

Solution of ψ_j's satisfying (2.1) to (2.5) is given by

$$\psi_j(x,y) = \pm c \int_0^\infty \frac{\exp(\mp k_y)}{k(1+M'k^2)-L} \cos(kx)\, dk\ (j = 1,2) \qquad (2.10)$$

where the path of integration is indented below the pole at $k = k_0$ to account for the outgoing nature of ψ_j's as $x \to \infty$ and c is a constant to be chosen such that (2.9) is satisfied. We may note that ψ_j's given by (2.10) remain finite as $(x^2 + y^2)^{1/2} \to 0$ so long as $T > 0$. However, for $T = 0$, (2.10) exhibit a logarithmic singularity as $r \to 0$ (cf. Yu and Ursell[4]) which accounts for a source/sink type behaviour at the shore-line in the absence of interfacial tension as stated in the introduction.

Alternative representation for $\psi_j (x, y)$'s are given by

$$\psi_j (x, y) = \pm c\left[\frac{\pi i}{1+3M\ k_0^2} \exp (\mp k_0 y + ik_0 x) + \int_0^\infty \frac{\exp(-kx)}{k^2 (1-M'k^2)^2 + L^2} \{k (1-M'k^2) \cos ky \mp L \sin ky\}\, dk\right]. \quad ...(2.11)$$

To satisfy (2.9), we must choose

$$c = \frac{i}{\pi}(1 + 3M'\ k_0^2). \quad ...(2.12)$$

Thus we obtain finally,

$$\Phi_j (x, y, t) = \pm \exp(\mp k_0 y) \cos (k_0 x + \sigma t) \pm \frac{\sin \sigma t}{\pi} (1 + 3M'\ k_0^2) \int_0^\infty \frac{\exp(-kx) \{k (1-M'\ k^2) \cos ky \mp L \sin ky\}}{k^2 (1 - M'\ k^2)^2 + L^2}\, dk. \quad ...(2.13)$$

In the absence of interfacial tension (2.13) reduces to

$$\Phi_j (x, y, t) = \pm \exp(\mp Ly) \cos (Lx + \sigma t) \pm \frac{\sin \sigma t}{\pi} \int_0^\infty \frac{\exp(-kx)}{k^2 + L^2} \{k \cos ky \mp L \sin ky\}\, dk. \quad ...(2.14)$$

Putting $s = 0$ in the expression for Φ_1 in (2.14), the potential function obtained by Stoker[2,3] is recovered. Again, putting $s = 0$ in Φ_1 into (2.13) we obtain

$$\Phi_1 (x, y, t) = \exp(-k_0 y) \cos (k_0 x + \sigma t) + \frac{\sin \sigma t}{\pi} (1 + 3Mk_0^2) \int_0^\infty \frac{\exp(-kx)}{k^2 (1 - Mk^2)^2 + K^2} \{k (1 - Mk^2) \cos ky - K \sin ky\}\, dk$$

which was derived by Packham[1] using a different approach.

4. Discussion

Potential functions representing incoming progressive waves against a vertical cliff in a two-fluid medium are obtained. Putting $s = 0$, the result for a deep ocean is recovered, more specifically, Φ_1 reduces to Stoker's[2,3] result when the surface tension is ignored and to Packham's[1] result when this is included. Also the potential functions in a two-fluid medium in the absence of interfacial tension are derived by simply putting $T = 0$. This problem can be further generalised to include the cases where the lower fluid is of uniform finite depth and/or the upper fluid is bounded by a horizontal rigid lid or a free surface.

Acknowledgement

The author takes this opportunity to thank Dr B. N. Mandal, Department of Applied Mathematics, Calcutta University, for his help and guidance in the preparation of this paper and the referee for his comments in revising the paper into the present form. This research is supported by the UGC project No. F-8-8/85 (SR-III).

References

1. B. A. Packham, *Proc. Camb. phil. Soc.* **64** (1968), 827.
2. J. J. Stoker, *Quart. appl. Math.* **5** (1947), 1.
3. I. J. Stoker, *Water Waves*. Interscience, New York, 1957.
4. Y. S. Yu and F. Ursell, *J. Fluid Mech.* **11** (1961), 529.

SUGGESTIONS TO CONTRIBUTORS

The INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS is devoted primarily to original research in pure and applied mathematics.

Manuscripts should be typewritten, double-spaced with sufficient margins (including abstracts, references, etc.) on one side of durable white paper. The initial page should contain the titel followed by author's name and full mailing address. The text should include only as much as is needed to provide a background for the particular material covered. Manuscripts should be submitted in triplicate.

The author should provide a short abstract, in triplicate, not exceeding 250 words, summarizing the highlights of the principal findings covered in the paper and the scope of research.

References should be cited in the text by the arabic numbers in superior. List of references should be arranged in the arabic numbers, author's name, abbreviation of Journal, Volume number (Year) page number, as in the sample citation given below :

For Periodicals

1. R. H. Fox, *Fund. Math.* 34 (1947) 278.

For Books

2. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, (1973) p. 283.

Abbreviations for the titles of the periodicals should, in general, conform to the *World List of Scientific Periodicals*.

All mathematical expressions should be written clearly including the distinction between capital and small letters. Clear distinction between upper and lower cases of c,p,k,z,s, should be made while writing the expression in hand. Also distinguish between the letters such as 'Oh' and 'zero'; l(el) and 1 (one); v, V and ν (Greek nu); r and γ (Greek gamma); x, X and χ (Greek chi); k, K and κ (Greek kappa); Greek letter lambda (Λ) and symbol for vector product ($\wedge$); Greek letter epsilon (ϵ) and symbol for 'is an element of' ($\in$). The equation numbers are to be placed at the right-hand side of the page. The name of the Greek letter or symbol should be written in the margin the first time it is used. Superscripts and subscripts should be simple and should be placed accurately.

Line drawings should be made with India ink on white drawing paper or tracing paper. Letterings should be clear and large. Photographic prints should be glossy with strong contrast. All illustrations must be numbered consecutively in the order in which they are mentioned in the text and should be referred to as Fig. or Figs. Legends to figures should be typed on a separate sheet and attached at the end of the manuscript.

Tables should be typed separately from the text and placed at the end of the manuscript. Table headings should be short but clearly descriptive.

Proofs should be corrected immediately on receipt and returned to the Editor. If a large number of corrections are made in the proof, the author should pay towards composition charges. In case, the author desires to withdraw his paper, he should pay towards the composition charges, if the same is already done.

For each paper, the authors will receive 50 reprints free of cost. Order for extra reprints should be sent with corrected page proofs.

Manuscripts, in triplicate, should be submitted to the Editor of Publications, *Indian Journal of Pure and Applied Mathematics*, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002 (India).

RN 19797/70 Regd. No. D-(C)-774

INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

No. 3 March 1989 Volume 20

CONTENTS

Published and Printed by Dr O. N. Kaul, Executive Secretary, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002, at Leipzig Press, D-52, N.D.S.E. Part I New Delhi 110049, Ph. 622490